OPTIMIZING RETAIL LOCATION: AN INTEGER LINEAR PROGRAMMING APPROACH

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ABSTRACT

We develop a spatial choice model to optimize location decisions for a retail chain. Existing location models are reviewed and are the framework upon which our model is built. The effects of market expansion and market cannibalization are discussed and are integral to the model developed. We assume that the retail chain already has a presence in the market area and endeavor to maximize sales for its new and existing stores. By representing customer demand as a nondecreasing concave function of the utility each customer associates with the facilities, a nonlinear programming problem is formulated. The model is then approximated by a piecewise linear scheme for the objective function. This enables an \( \alpha \)-optimal solution to be derived, using a simplex method linear programming algorithm.
DEDICATION

This thesis is dedicated to my family. It would not have been possible to complete this thesis without their endless support and encouragement.
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### LIST OF SYMBOLS

- \( N \): set of demand points, \( N = 1, 2, ..., n \)
- \( F \): existing facility locations and potential new sites, \( F \subseteq N \)
- \( E \): existing facilities, \( E \subseteq F \)
- \( S \): existing facilities owned by the company (sister stores), \( S \subseteq E \)
- \( C \): existing facilities owned by the competition, \( C \subseteq E \)
- \( P \): possible sites for new stores, \( P = F - E \)
- \( \hat{P} \): set of locations for up to \( m \) new stores, \( \hat{P} \subseteq P \)
- \( \hat{S} \): the company’s existing facilities and \( m \) new stores, \( \hat{S} = S \cup \hat{P} \)
- \( w_i \): market potential in dollars for each demand point \( i \in N \)
- \( A_j \): the attractiveness of facility \( j \)
- \( \alpha \): parameter weighting attractiveness
- \( d_{ij} \): distance between customers at demand point \( i \) and facility \( j \)
- \( \beta \): parameter weighting distance sensitivity
- \( u_{ij} \): utility of customers at demand point \( i \) for facility \( j \)
- \( U_i \): total utility for customers at demand point \( i \)
- \( U_i(S) \): total utility for customers at demand point \( i \) from \( S \)
- \( U_i(\hat{P}) \): total utility for customers at demand point \( i \) from \( \hat{P} \)
- \( U_i(C) \): total utility for customers at demand point \( i \) from \( C \)
- \( U_i(\hat{S}) \): total utility for customers at demand point \( i \) from \( \hat{S} \)
- \( MS_{ij} \): probability of customers at demand point \( i \) using facility \( j \) out of all available facilities, new and existing
\( MS_i \)  \( g_i \)  \( \lambda_i \)

- \( MS_i \): total market share for demand point \( i \)
- \( g_i \): variable demand function for customers at demand point \( i \) in \( N \)
- \( \lambda_i \): elasticity parameter for demand point \( i \)
1 INTRODUCTION

In today’s marketplace strategic planning is essential to the success of a retail chain. Successful business decisions are more and more often based on the contributions of market research analysts, rather than pure instinct or business savvy. Analysts assess the current marketing environment, examining how factors such as location and size are affecting the performance of current stores. Next, future changes in the marketplace are anticipated. These changes may come from competitors attempting to improve their own performance, thus, making reactive strategies necessary. The changes may also be demographic. Retailers must foresee changes in economic conditions and the ever evolving preferences and lifestyles of their customers [1].

However, the most crucial decision that a retailer will make is the selection of store locations. According to Jain and Mahajan, a retailer’s location is one advantage that cannot easily be duplicated [8]. Other competitive strategies can be imitated, such as pricing, merchandising and even marketing, but it is difficult to match a good location. There are also significant expenses incurred in acquiring a location and building a new store. Thus, once a location has been selected, management can no longer modify the locational strategy without considerable financial impact [3]. It is for these reasons that locational strategy has become a major component of retail planning [6].

In Section 2, this paper will review the development of several location algorithms leading up to the competitive facility location model proposed by Aboolian, Berman and Krass. The model proposed by Aboolian, et al will be examined and several modifications proposed. We show the necessity of these modifications in Section 3 and develop the Multi-Facility Retail Location Model. The Tangent Line Approximation Procedure developed by Aboolian et al will be discussed and the
suitability of this procedure for approximating our algorithm discussed. This will be followed by a demonstration of our algorithm using actual data from a retail chain. We then explore the ramifications of the solution generated for the given market area. The optimality of the solution generated by our algorithm will be validated by comparing it to the location strategy proposed by the company’s research analyst. Section 6 will discuss further modifications to our model and directions for future research.
2 DEVELOPMENT OF SITE SELECTION MODELS

Several approaches have been proposed for the site selection of new stores. The most commonly used methodology is the analog model. In the analog model, an existing store similar to the proposed store is identified and then the power of the analog store to draw customers from various areas is observed. This information is used to estimate the trading area and sales for the proposed store. The analog method is applied to the other store sites under consideration and then the best site is chosen. Although the analog model seems easy to implement, it does not take into account the competitive environment when evaluating the various store sites. Further, the results may often vary, depending on which analog stores are selected for examination. Human error may also be introduced, as the accuracy of the analog model often depends on the judgment of the analyst [3].

The gravitational and spacial choice model was developed by David Huff to meet some of the shortcomings of the analog model [7]. Huff’s model was based on the work of Robert Duncan Luce and his choice axiom [9]. Luce’s choice axiom states

\[ P_{ij} = \frac{w_i}{\sum_j w_j}. \]  

(1)

Luce maintained that the probability of an individual selecting one item over another from a collection of many items is not affected by the other items in the collection. Instead, the probability is given as the ratio of a particular item’s weight, \( w_i \), to the total weight of the pool of items [9]. Huff adapted Luce’s axiom and said that the probability of an individual choosing a particular retail store is equal to the ratio of the utility of that store to the sum of the utilities of all other stores under consideration [7]. Huff was specifically interested in the size of the store and the
distance between the consumer and the store [7]. Huff’s model is

\[ P_{ij} = \frac{S_j}{\sum_{k=1}^{n} \frac{S_k}{D_{ik}}} \]  

where \( P_{ij} \) is the probability of a person at zone \( i = 1, 2, \ldots, m \) traveling to a particular shopping facility \( j \). The size of the shopping facility \( k \) is represented by \( S_k \) and the distance or travel time between zone \( i \) to facility \( j \) is \( D_{ik} \). The effect of distance or travel time on the probability of shopping at a particular store is given as \( \beta \), which is an empirically estimated parameter. The number of stores under consideration in the region is \( n \).

Nakanishi and Cooper built upon Huff’s model and proposed the multiplicative competitive interaction (MCI) model [10]. In this formulation other store characteristics are considered, along with size and distance. Some characteristics that may be included in the MCI model are the number of checkouts, number of aisles and even the location at certain intersections. This model then estimates the market share of a proposed store and simulates the effect that modifications in store characteristics may have on the performance of the proposed stores.

Eventually, retail chains saw the benefits of locating multiple stores to establish a network of stores, instead of merely opening one new location at a time. By establishing a network of stores in a trade area, retail chains realized they could reduce competitive encroachment and also reduce future cannibalization of their own stores. Since the aforementioned models only evaluate sites for the addition of a single store, Avijit Ghosh and Samuel Craig proposed an approach to site selection based on location allocation modeling to solve multiple facility location problems [6]. Instead of merely locating a single store, the competitive equilibrium model focused on meeting the needs of changing retail strategy and introduced simultaneous selection of store locations within the framework of a spatial choice model [6]. The
main component of this store location procedure is an estimate of the expected market share of a proposed store, given as

\[ P_{ij} = \frac{\prod_{e=1}^{E} A_{ij}^{\beta_e}}{\sum_{k=1}^{n} \prod_{e=1}^{E} A_{ik}^{e}} \]  

(3)

where

\[ P_{ij} = \text{the probability of an individual at zone } i \text{ shopping at store } j; \]
\[ A_{ike} = \text{the } e^{th} \text{ attribute describing store } k, \ (k = 1, 2, \ldots, n) \]
\[ \text{by customers in zone } i; \]
\[ \beta_e = \text{the parameter associated with attribute } e, \ (e = 1, 2, \ldots, m) \]
\[ \text{affecting store choice.} \]

The competitive facility location model presented by Aboolian, Berman and Krass endeavors to address the marketing concepts of cannibalization and market expansion. Cannibalization takes place when a company, in effect, eats its own market. Even though new stores may generate considerable revenue, they may undermine the profitability of existing stores. Even though some cannibalization is unavoidable when a company expands its market presence, it is often counteracted by the market expansion effect. Market expansion takes place when consumer demand increases as a result of the new facilities. The new locations may attract customers that were previously underserviced by the company or customers may find the new facilities more convenient leading to increased spending.

Similar to Ghosh and Craig, Aboolian et al use the general framework of a spatial interaction model in order to represent market expansion and cannibalization [1]. In
fact, the estimate of expected market share for a proposed store, given as

\[ M_{S_{ij}} = \frac{u_{ij}}{\sum_{k \in E \cup S} u_{ik}} \] (4)

\[ u_{ij} = \frac{A_j^\alpha}{d_{ij}^\beta} \] (5)

is very similar to equation (3) in the competitive equilibrium model [1]. As in the earlier model developed by Ghosh and Craig, \( A_j \) represents the attractiveness of facility \( j \), and is weighted by the parameter \( \alpha \). The denominator, \( d_{ij} \), is weighted by the parameter \( \beta \) and measures the distance from customers in zone \( i \) to facility \( j \). Aboolian et al. then created a demand function \( g_i \) for customers at location \( i \) based on the standard economic assumption that customer demand is concave and non-decreasing [1]. Their demand function is

\[ g_i(U'_i(S)) = 1 - \exp(-\lambda_i U'_i(S)) \] (6)

\[ U'_i(S) = \sum_{j \in E \cup S} u_{ij} \] (7)

for \( E \), existing stores, and \( S \), the set of locations for \( m \) new stores. Finally, [1] formulates the problem as a nonlinear Knapsack problem

\[ \max_{S \subseteq \bar{E} | S| = m} \sum_{i \in N} \sum_{j \in S} w_i g_i M_{S_{ij}} \] (8)

where \( \bar{E} \) is the set of locations available for new stores and \( w_i \) is the number of customers in zone \( i \). Berman and Krass refer to the above model as the competitive form of the model since the objective function only maximizes expenditures for the \( m \) new stores entering the market.

In the monopolistic case, the objective function maximizes expenditures for both new and existing stores. Recall that \( M_{S_{ij}} = \frac{u_{ij}}{\sum_{k \in E \cup S} u_{ik}} \), thus, \( \sum_{j \in E \cup S} M_{S_{ij}} = 1 \) for
\( i \in N \). Equation (8) can now be simplified to

\[
\max_{E \cup S, |S| = m} \sum_{i \in N} w_i g_i .
\] (9)

In the next section, we will develop our model by expanding upon equations (8) and (9).
The model we propose, hereafter referred to as the Multi-Facility Retail Location Model, follows the same basic premise as other spatial interaction models. By combining the constructs of Aboolian’s Competitive Facility Location Model and the monopolistic model of Berman and Krass we endeavor to create a model that will adapt to variable market scenarios. We do this by making several important modifications to equations 8 and 9.

First, we advocate a model that will simultaneously consider the effects of competitors and of cannibalization. Instead of focusing on a situation that is strictly monopolistic or competitive, we desire a more flexible model. For instance, consider a scenario in which a company and its competitors already have existing facilities. In this case, it is insufficient to consider only the sales generated by the new facilities, since there will likely be cannibalization of sales from the company’s existing facilities. While, the monopolistic model allows for cannibalization of sales, it too is lacking, since it does not consider the competitor’s existing locations.

The model we propose differentiates between the company’s existing stores and existing competitive stores. Clearly, most companies are interested in how potential new stores will affect their existing stores and then what effect, if any, the competition might have on these new stores. Therefore, we reference competitive stores as $C$, sister stores as $S$, locations available for new stores as $P$ and the final set of new stores as $\hat{P}$ where the maximum number of new stores to be located is $m$. Thus, our model includes both existing stores and proposed stores in the objective function. We maintain that this is necessary in order to address the effect of market cannibalization in our model. Ultimately, retail chains consider two things when locating new stores, the increased revenue that new locations will provide, and the cannibalization effect that a new location may have on its sister stores. Therefore,
the objective function of our model focuses on maximizing the revenue with respect to all of the company’s stores rather than just the new locations.

We will also replace Aboolian’s customer variable $w_i$ with a market potential variable. Recall that the Competitive Facility Location Model used $w_i$ to represent the number of customers in zone $i$. We maintain that retail chains are more interested in focusing on potential customer expenditures rather than just the number of customers in a region. Reinartz and Kumar and support this idea stating

*the market potential of a store is by far the most important driver of store sales performance and sales productivity performance* [11].

To demonstrate the idea of market potential, consider two very different neighborhoods, a small, yet upscale suburb and a more densely populated urban area with much lower incomes. Depending on the type of product or service offered by the facilities, the smaller suburb may prove to be more profitable to the retail chain if the suburb has higher customer expenditures. We therefore, introduce the idea of a market potential variable. This variable can be calculated by examining annual customer expenditures on the types of products relevant to the retail chain for each demand point $i$. For example, a chain selling ski equipment would undoubtedly have a much higher market potential in the northern United States near ski resorts, than in say, Florida. Thus, by focusing on the market potential, the model we will develop enables a company to base locational decisions on where the potential customers are, rather than where the most people are.

3.1 Model Formulation

Since our model follows the customary structure of spatial interactions models with variable spending, we represent customer utility $u_{ij}$, the affinity that customers
at $i$ have for facility $j$, as

$$u_{ij} = \frac{A_j^\alpha}{(d_{ij} + 1)^\beta}. \quad (10)$$

Recall that $\alpha$ is a parameter weighting the attractiveness, $A$, of facility $j$ for demand point $i$ and $\beta$ is a parameter that weights the distance, $d_{ij}$. This is a slight departure from Huff’s model which uses $u_{ij} = A_j^\alpha d_{ij}^{-\beta}$. Huff’s model assumes that if customers are sufficiently close to a facility $j$ that the facility will capture all customers, no matter how unattractive the facility is. This is unrealistic, since customer data is quite often aggregated and customer point $i$ may represent a significantly large region [5]. The attractiveness variable $A_j$ is dependent on such attributes as number of checkouts, facility size, signage, etc. Next, the total utility for demand point $i$ is defined as

$$U_i = \sum_{j \in E \cup \hat{P}} u_{ij}. \quad (11)$$

Obviously, businesses are interested in the market share being captured by their stores, thus, the market share for a specific store and demand point, $MS_{ij}$, is represented by

$$MS_{ij} = \frac{u_{ij}}{U_i}. \quad (12)$$

Then by letting,

$$U_i(S) = \sum_{j \in S} u_{ij}, \quad U_i(\hat{P}) = \sum_{j \in \hat{P}} u_{ij} \quad \text{and} \quad U_i(C) = \sum_{j \in C} u_{ij}$$

the company’s total market share for demand point $i$ can be given by

$$MS_i = \frac{U_i(S) + U_i(\hat{P})}{U_i(S) + U_i(\hat{P}) + U_i(C)} \quad (13)$$

$$= \frac{U_i(S)}{U_i}. \quad (14)$$
Note that if we let $\hat{S} = S \cup \hat{P}$ we have,

$$U_i(S) + U_i(\hat{P}) = U_i(\hat{S}) = \sum_{j \in \hat{S}} u_{ij}.$$ 

Thus $U_i(\hat{S})$ represents the sum of utilities for existing and new stores owned by the company. To simplify the notation needed later, we can now write

$$MS_i = \frac{U_i(\hat{S})}{U_i} = 1 - \frac{C}{U_i} \quad (15)$$

where $C$ denotes $U_i(C)$.

We next incorporate a demand function. Similar to Berman and Krass [4], the total demand of customers at demand point $i \in N$ is assumed to be elastic and is a function of $U_i$. We represent demand with a concave, nondecreasing function with range $[0, 1]$. Obviously there are many functions that fit this criteria, we designate our demand function to be $g_i(U_i)$, which will be specifically define later. Since we want to maximize the share of the market captured by the company’s facilities, both existing and new sites, we must take into consideration the market size being captured. The objective function used by Aboolian et al considers the number of customers $w_i$ in a particular location. As previously mentioned, we will replace this with a variable representing market potential, while using the same notation, $w_i$. For the sake of clarity, we will assume that $w_i$ is the annual customer revenue relevant to the products the company carries. We can then state the objective function of our problem as:

$$\max_{\hat{P} \leq m} \sum_{i \in N} \sum_{j \in \hat{S}} w_i g_i(U_i) MS_{ij} \quad (16)$$

Next, we denote

$$V_i(U) = g_i(U_i) MS_{ij} \quad (17)$$
Thus, \( V_i(U_i) \) represents the total customer demand in market \( i \) captured by the company’s stores, new and existing. We now simplify the objective function as follows:

\[
Z(\hat{S}) = \sum_{i \in N} w_i V_i(U_i) .
\]  

(18)

This paper will focus on a specific case of (18) where \( V_i(U) \) is concave and non-decreasing.

**Assumption 1.** Let \( H_i = \sum_{j \in E} u_{ij} \) Then the total demand function \( V_i(U) \) is a concave nondecreasing function of \( U \) in \([H_i, \infty)\) for all \( i \in N \). This is satisfied when the expenditure function \( g_i(U) \) is concave, nondecreasing and vanishes at 0.

**Theorem 1.** Suppose that \( g_i(U) \) is a twice differentiable non-decreasing concave function such that \( g_i(U) = 0 \) for some \( i \in N \). Then \( V_i(U) \) satisfies Assumption 1.

**Proof.** We first simplify the notation by dropping the subscript. By (15) and (17),

\[
V(U) = g(U) \left( 1 - \frac{C}{U} \right).
\]

Taking the derivative with respect to \( U \), yields

\[
V'(U) = g'(U) \left( 1 - \frac{C}{U} \right) + g(U) \frac{C}{U^2} \geq 0 ,
\]

since \( g(U) \) is non-decreasing. We now take the second derivative, with respect to \( U \).

\[
\begin{align*}
V''(U) &= g''(U) \left( 1 - \frac{C}{U} \right) + g'(U) \frac{C}{U^2} + g(U) \left( \frac{C}{U^3} \right) \\
V''(U) &= g''(U) \left( 1 - \frac{C}{U} \right) + g'(U) \frac{2C}{U^2} - g(U) \frac{C}{U^3}
\end{align*}
\]

By the concavity of \( g(U) \), we have \( g''(U) \leq 0 \) and the quantity \( 1 - \frac{C}{H} \geq 0 \), so the
first term in the above equation is nonpositive. Consider
\[ g'(U) \frac{2C}{U^2} - g(U) \frac{C}{U^3} = 2C \frac{g'(U)}{U^3} (U - g(U)). \]
The derivative of
\[ g'(U)U - g(U) = g''(U)U \leq 0. \]
Then, since \( \frac{2C}{U^2} \) is positive,
\[ g'(U) \frac{2C}{U^2} - g(U) \frac{C}{U^3} \leq 0. \]
Therefore, given that \( V'(\tilde{U}) \geq 0 \) and \( V''(\tilde{U}) \leq 0 \),
\( V(\tilde{U}) \) is concave nondecreasing.

We adopt the exponential demand function used by Aboolian et al and define
\[ g_i(U) = 1 - \exp(-\lambda_i U_i) . \]
(19)
Here, \( \lambda_i > 0 \) is the exponential decay parameter, or the rate at which the total demand of a customer in market \( i \) reaches its saturation point \( w_i \). When the elasticity parameter is high, the demand is inelastic. In other words, the customer expenditures will be close to the market potential regardless of the utility of the particular facilities. When \( \lambda_i \) is close to 0, the demand will be extremely elastic and a small change in the utility of the facility will result in a large change in customer expenditures. Section 5 will further examine the effects of varying the demand parameter.

We now use the previous results to formulate our model as an optimization problem. The variable \( x_j \) is binary, where \( x_j = 1 \) if a facility is to be located at \( j \).
and 0 otherwise.

\[
M_0 \begin{cases} 
\max & \sum_{i \in N} w_i g_i \left( \sum_{j \in E} u_{ij} + \sum_{j \in P} u_{ij} x_j \right) \left( 1 - \frac{\sum_{j \in C} u_{ij}}{\sum_{j \in E} u_{ij} + \sum_{j \in P} u_{ij} x_j} \right) \\
\text{s.t.} & \sum_{j \in P} x_j \leq m, \\
& x_j \in \{0, 1\}, \quad j \in P
\end{cases}
\tag{20}
\]

The above model will be referred to as \( M_0 \) and is a nonlinear integer program. If we suppose that \( \hat{x} = (x_{i1}, x_{i2}, \ldots, x_{i\hat{P}}) \) is a feasible solution to \( M_0 \), then we can write

\[
U_i(\hat{x}) = \sum_{j \in S} u_{ij} x_j.
\tag{21}
\]

Thus, we can see from (20) and (21) that the objective function is the sum of \( n \) concave nondecreasing functions composed with linear functions.
The TLA procedure approximates a class of concave integer programming models. This class of models is referred to as $D_0$.

\[
D_0 \begin{cases} 
\max F(x) = \sum_{i=1}^{n} f_i(\phi_i(x)) \\
s.t A(x) \leq b \\
x \in \mathbb{R}^m \quad x, \text{ integer}
\end{cases}
\] (22)

In the model above $f_i(\phi_i(x))$ are composite functions where $f_i(y)$ is a twice differentiable, nondecreasing concave function of $y \in \mathbb{R}^+$. The linear functional $\phi_i(x)$ is equivalent to $C_i^T x = (c_{i1}, c_{i2} \ldots c_{im})^T (x_1, x_2 \ldots, x_m)$ (23)

where $i = 1 \ldots n, c_{ij} \geq 0$. Here we assume that $0 \leq \phi_i(x) \leq \overline{\phi_i(x)}$. Thus, there could be a different upperbound for each $\phi_i(x)$. The inequality $A(x) \leq b$ is a system of linear constraints.

Since each term in the objective function of $D_0$ is a composition of a concave function with a linear function, we see that $M_0$ is a special case of this programming model. As a nonlinear integer programming model, solutions are difficult to generate and often do not reach optimality [4]. Therefore, we will use a linear program to approximate $D_0$, and ultimately apply the results to $M_0$. Then each $f_i(\phi_i(x))$ is replaced by a concave piecewise linear function, $f_i^A(\phi_i(x))$, where $f_i^A(\phi_i(x))$ bounds $f_i(\phi_i(x))$ from above. So for $i = 1 \ldots n$, we have

\[
f_i^A(\phi_i(x)) \geq f_i(\phi_i(x)) \geq 0 \text{ for } 0 \leq \phi \leq \overline{\phi_i}.
\] (24)

We know that $f_i^A(\phi_i(x))$ exists because $f_i(\phi_i(x))$ is bounded on the domain $[0, \overline{\phi_i}]$. 
Therefore, the domain of \( f_i^A(\phi_i(x)) \) is \( 0 \leq \phi \leq \phi_i \). Note that \( \phi_i \) is dependent on \( x \) and \( f_i \) dependent on \( \phi_i \). Thus, we will often refer to \( f_i(\phi_i(x)) \) as simply \( f_i(\phi_i) \) to simplify the notation. Let

\[
F(x) = \sum_{i=1}^{n} f_i(\phi_i) \quad \text{and} \quad F^A(\phi_i) = \sum_{i=1}^{n} f_i^A(\phi_i).
\]  

(25)

We can now conclude the following.

**Lemma 1.**

\[
\alpha_i = \max_{0 \leq \phi_i \leq \overline{\phi}} \left\{ \frac{f_i^A(\phi_i) - f_i(\phi_i)}{f_i(\phi_i)} \right\}
\]

(26)

be the maximum relative error for approximating \( f_i(\phi_i) \) by \( f_i^A(\phi_i) \). Define

\[
\alpha' = \max_{i \in N} \{ \alpha_i \}.
\]

(27)

Thus, for any feasible solution \( x \) of \( D_0 \) and each \( \phi_i(x) \)

\[
F(\phi_i(x)) \leq F^A(\phi_i(x)) \leq (1 + \alpha') F(\phi_i(x)).
\]

(28)

**Proof.** We will consider a single function \( \phi_i(x) \) and therefore drop the subscript \( i \). By definition \( F(\phi) = \sum_{i=1}^{n} f_i(\phi) \) and \( F^A(\phi) = \sum_{i=1}^{n} f_i^A(\phi) \). Then, for all \( i \),

\[
\sum f_i(\phi) \leq \sum f_i^A(\phi)
\]

by equation 24 and we have \( F(\phi) \leq F^A(\phi) \). Using equations 26 and 27 we obtain

\[
\alpha' \geq \frac{f_i^A(\phi) - f_i(\phi)}{f_i(\phi)}
\]

\[
f_i(\phi) \alpha' \geq f_i^A(\phi)
\]

\[
f_i(\phi) \alpha' + f_i(\phi) \geq f_i^A(\phi)
\]

\[
(1 + \alpha') f_i(\phi) \geq f_i^A(\phi).
\]
Thus, for any feasible solution $x$ of $D_0$ we have

$$F(\phi(x)) \leq F^A(\phi(x)) \leq (1 + \alpha')F(\phi(x)).$$

\[\square\]

Our goal is to specify an error bound $\alpha$ and then to construct concave piecewise linear functions $f^\alpha_i(\phi_i)$ that belong to the family of functions $f_i(\phi_i)^A$ such that 24 is satisfied and $\alpha_i \leq \alpha$ by 26. If we let

$$F^\alpha(x) = \sum_{i=1}^{n} f^\alpha_i(\phi_i). \quad (29)$$

we can then construct a second programming model based on $D_0$. We will refer to this model as $D_\alpha$.

$$D_\alpha \begin{cases} 
\max F^\alpha(x) = \sum_{i=1}^{n} f^\alpha_i(\phi_i(x)) \\
\text{s.t} A(x) \leq b \\
x \in \mathbb{R}^m \quad x, \text{ integer} 
\end{cases}$$

Since $F^\alpha(x)$ is piecewise linear, it can be formulated as a linear integer program. Thus, we will be able to find an optimal solution to $D_0$ with relative error $\alpha$.

4.1 Construction of Piecewise Linear Over-Approximator Functions

In this section we will show how to construct a piecewise linear function for a specific $i$. To simplify notation we again drop the subscript.

$$f^\alpha(\phi) \geq f(\phi) \quad \phi \in [0, \bar{\phi}] \quad (30)$$

$$\max_{0 \leq \phi \leq \bar{\phi}} \left\{ \frac{f^\alpha(\phi) - f(\phi)}{f(\phi)} \right\} \leq \alpha. \quad (31)$$
We will show that for a piecewise linear function satisfying equation 30, it is sufficient to show 31 is satisfied at the beginning or end of each of the line segments of the function.

**Theorem 2.** Let \( f^A(\phi) \) be a non-decreasing concave piecewise linear function with domain \([0, \phi]\) such that \( f^A(\phi) \geq f(\phi) \) for \( \phi \in [0, \phi] \). Let

\[
\epsilon(\phi) = \frac{f^A(\phi) - f(\phi)}{f(\phi)}
\]

be the relative error between \( f^A(\phi) \) and \( f(\phi) \). Then, \( \epsilon(\phi) \) is maximized at the beginning of one of the line segments of the approximating function, \( f^A(\phi) \).

**Proof.** Let the \( l^{th} \) segment defining \( f^A(\phi) \) have endpoints \( c_l \) and \( c_{l+1} \), with slope \( b_l \). We will show that

\[
\max_{\phi \in [c_l, c_{l+1}]} \frac{f^A(\phi)}{f(\phi)} \in \left\{ \frac{f^A(c_l)}{f(c_l)}, \frac{f^A(c_{l+1})}{f(c_{l+1})} \right\}.
\]

If \( f(\phi) \) is constant on \([c_l, c_{l+1}]\), the conclusion is trivial. Therefore, without loss of
generality, we will assume that $f(\phi)$ is non-constant on the interval $[c_l, c_{l+1}]$. Then for $\phi \in [c_l, c_{l+1}]$ let

$$S(\phi) = f(c_l) + (\phi - c_l) \frac{f(c_{l+1}) - f(c_l)}{c_{l+1} - c_l}$$

(34)

be a straight line that connects $(c_l, f(c_l))$ and $(c_{l+1}, f(c_{l+1}))$. Then, the slope of this line is

$$s_l = \frac{f(c_{l+1}) - f(c_l)}{c_{l+1} - c_l}.$$  

(35)

Since we have assumed that $f(\phi)$ is concave and non-constant on $[c_l, c_{l+1}]$, then, $s_l > 0$ and

$$S(\phi) = f(\phi) \text{ for } \phi \in \{c_l, c_{l+1}\} \text{ and } S(\phi) \leq f(\phi) \text{ for } \phi \in (c_l, c_{l+1}) .$$  

(36)

Thus, we see that

$$\frac{f^A(\phi)}{S(\phi)} \geq \frac{f^A(\phi)}{f(\phi)} \text{ for } \phi \in [c_l, c_{l+1}]$$

(37)

with equality occurring at the endpoints, when $\phi = c_l$ and $\phi = c_{l+1}$. Next, let $r = \frac{b}{s_l}$.
By definition of $\epsilon(\phi)$,

$$
\epsilon(c_l) = \frac{f^A(c_l) - f(c_l)}{f(c_l)}.
$$

(38)

So for $\phi \in [c_l, c_{l+1}]$,

$$
f^A(\phi) = f^A(c_l) + b_l(\phi - c_l)
$$

(39)

and since $b_l = rs_l$

$$
f^A(\phi) = f^A(c_l) + rs_l(\phi - c_l)
$$

$$
= f(c_l) + f^A(c_l) - f(c_l) + rs_l(\phi - c_l)
$$

$$
= (f(c_l) + (f^A(c_l) - f(c_l))) \frac{f(c_l)}{f(c_l)} + rs_l(\phi - c_l)
$$

$$
= (1 + \epsilon(c_l))f(c_l) + rs_l(\phi - c_l)
$$

$$
= (1 + \epsilon(c_l))f(c_l) + (1 + \epsilon(c_l))(s_l(\phi - c_l)) + rs_l(\phi - c_l)
$$

$$
- (1 + \epsilon(c_l))s_l(\phi - c_l)
$$

$$
= (1 + \epsilon(c_l))(f(c_l) + s_l(\phi - c_l)) + [r - (1 + \epsilon(c_l))]s_l(\phi - c_l)
$$

$$
\frac{f^A(\phi)}{S(\phi)} = \frac{(1 + \epsilon(c_l))(f(c_l) + s_l(\phi - c_l)) + [r - (1 + \epsilon(c_l))]s_l(\phi - c_l)}{f(c_l) + s_l(\phi - c_l)}
$$

$$
= 1 + \epsilon(c_l) + [r - (1 + \epsilon(c_l))] \frac{s_l(\phi - c_l)}{f(c_l) + s_l(\phi - c_l)}
$$

We see that $\frac{s_l(\phi - c_l)}{f(c_l) + s_l(\phi - c_l)}$ is increasing since the derivative is always positive. This implies that $\frac{f^A(\phi)}{S(\phi)}$ is nondecreasing when $r \geq 1 + \epsilon(c_l)$ and is decreasing otherwise. Either way, $\frac{f^A(\phi)}{S(\phi)}$ is maximized at one of the endpoints and since $\frac{f^A(\phi)}{S(\phi)} = \frac{f^A(\phi)}{f(\phi)}$ at the endpoints, we see that $\frac{w^A(\phi)}{w(\phi)}$ is maximized at $c_l$ or $c_{l+1}$. \hfill \Box

**Lemma 2.** Consider $c \in [0, \overline{c}]$ and $d \in R$ where $f(c) \leq d$. For a point $c_T \in [c, \overline{c}]$, define $b_L(c_T) = \frac{f(c_T) - d}{c_T - c}$ as the slope of the line from $(c, d)$ to $(c_T, f(c_T))$. Let $L(\phi) = d + b_L(c_T)(\phi - c)$ be the equation of that line, where $\phi \in [c, c_T]$ . Then:

1. There exists $c_T \in [c, \overline{c}]$ such that $f'(c_T) = b_L(c_T)$ ie. the slopes of $L(\phi)$ and
$f(\phi)$ coincide at $c_T$ if and only if

$$f(\bar{\phi}) \geq f'(\bar{\phi})(\bar{\phi} - c) + d.$$  \hspace{1cm} (40)

2. If condition 40 holds, then $c_T$ is unique and can be found by solving

$$f'(\phi)(\phi - c) + d - f(\phi) = 0$$  \hspace{1cm} (41)

for $\phi \in [c, \bar{\phi}]$.

**Proof.** Let $\phi \in [c, \bar{\phi}]$ and $T(\phi) = f'(\phi)(\phi - c) + d - f(\phi)$. Then

\[ T'(\phi) = f'(\phi) + f''(\phi)(\phi - c) - f'(\phi) \]
\[ T'(\phi) = f''(\phi)(\phi - c) \]

Since $f(\phi)$ is concave, $f''(\phi) < 0$. Thus, $T'(\bar{\phi}) < 0$ and $T(\phi)$ is strictly decreasing on $[c, \bar{\phi}]$. Recall the assumption that $d \geq f(c)$. If $d = f(c)$, then $T(c) = f'(c)(c - c) + d - f(c) = d - f(c) = 0$ and we have $c_T = c$ as the unique solution to $T(\phi)$. If $d > f(c)$, then $T(c) = d - f(c) > 0$ and $T(\phi)$ is decreasing on $[c, \bar{\phi}]$. Then $T(\phi) = 0$ has a unique solution if and only if equation 40 holds. Let $c_T$ be the root of $T(\phi) = 0$, then $b_L(c_T) = f'(c_T)$ by the definition of $b_L$. \hfill \square

**Lemma 3.** Consider $c \in [0, \bar{\phi}]$. Let $L(\phi) = w(c) + f'(\phi - c)$ be the line tangent to $f(\phi)$ at point $c$, for $\phi \in [0, \bar{\phi}]$. Then, for $\alpha > 0$, there exists $c_E \in [0, \bar{\phi}]$ such that

$$L(c_E) - f(c_E)(1 + \alpha) = 0$$  \hspace{1cm} (42)

if and only if

$$L(\bar{\phi}) \geq f(\bar{\phi})(1 + \alpha).$$  \hspace{1cm} (43)
Moreover, if a solution to equation (42) exists, then it is unique.

Proof. Let $T(\phi) = f(\phi)(1 + \alpha) - L(\phi)$. $T(\phi)$ is a concave function since $f(\phi)$ is concave. Therefore, $T(\phi)$ has at most one root on $[c, \bar{\phi}]$. If equation (43) holds, then $T(\bar{\phi}) \leq 0$ and such a real root exists. The proof of the converse is similar. \qed 

The following summarizes the basic steps of the TLA procedure.

1. Set $l = 1$, $c = c_1 = c_T = 0$ and $b_1 = f'(0)$

2. If (43) fails set $c_{l+1} = \bar{\phi}$, $L = l$ and STOP. Else, use the bisection method to find the unique root, $c_E$ of (42) and set $c_{l+1} = c_E$. If $c_{l+1} = \bar{\phi}$, set $L = l$ and STOP. Else proceed to step 3.

3. Set $l = l + 1$. If (40) holds proceed to step 4. Else, set $L = l$ and $c_L = \bar{\phi}$. If $f(c_L)(1 + \alpha) \leq f(\bar{\phi})$ then set $b_l = \frac{f(\bar{\phi}) - (1+\alpha)f(c_L)}{\bar{\phi} - c_L}$. Else, set $b_l = 0$ and STOP.

4. Use the bisection method to find the unique root, of 41. Set $c_T$ equal to that root and $b_l = f'(c_T)$.

5. If $c_T = \bar{\phi}$, set $c_{l+1} = c_T$, $L = l$ and STOP. Else, repeat step 2.

Figure 3: $f^\alpha(\phi)$ where $c_L = \bar{\phi}$ and $b_l = 0$
4.2 Using the TLA Procedure to Approximate the Multi-Facility Retail Location Model

Now that we’ve established the validity of the TLA procedure we can apply it to our location model $M_0$. Using the format set forth by Aboolian in [1] we can develop a linear approximating model for our original model $M_0$. Recall that $V_i(U)$ is the $i^{th}$ term of the objective function of $M_0$, thus for $i \in N$ we define $f_i^\alpha(U)$ as the $\alpha$-approximator of $V_i(U)$, where $U \in [0,U_i]$ and $\overline{U}_i = \sum_{j \in P} u_{ij}$. Thus, $w_i^\alpha(U)$ is constructed by the TLA procedure. Since $f_i^\alpha(U)$ is piecewise linear, we define $L_i$ to be the number of linear segments $l$. The endpoints of $l$ are $c_l$ and $c_{l+1}$ for each $l \in 1, \ldots, L_i$. We define the slope of segment $l$ as $b_l$, where the length of segment $l$ in terms of the horizontal axis is $a_l = c_{l+1} - c_l$. Thus, we can write the function $f_i^\alpha(U)$, $U \in [0,U_i]$ as

$$f_i^\alpha(U) = \sum_{l=1}^{L_i(U)} a_{li} b_{li} y_{li} \quad (44)$$

where

$$L_i(U) = \max\{l : c_l \leq U\}$$
and

\[ y_{li} = \begin{cases} 
1 & \text{if } l < L_i(U) \\
\frac{U-c_{li}}{a_{li}} & \text{if } l = L_i(U) .
\end{cases} \]

We can now approximate our linear program \( M_0 \) by the following linear integer program \( M_\alpha \).

\[
M_\alpha \left\{ \begin{array}{ll}
Z^\alpha & = \max \sum_{i \in N} \sum_{l=1}^{L_i} f_{il} a_{li} b_{li} y_{li} \\
\text{s.t.} & \sum_{j \in \hat{S}} u_{ij} x_j = \sum_{l=1}^{L_i} a_{li} y_{li} \text{ for } i \in N, \\
& \sum_{j \in P} x_j \leq m, \\
& 0 \leq y_{li} \leq 1 \text{ for } i \in N, l = 1, \ldots, L_i, \\
& x_j \in \{0, 1\} \text{ for } j \in P .
\end{array} \right.
\]

(45)

The SAS code given in Appendix 6 is formulated from the above results. We now demonstrate the Multi-Facility Retail Location algorithm and the TLA procedure with an application in Section 5.
The data we use to illustrate the effectiveness of the TLA procedure and the Multi-Facility Retail Location Model is taken from a retail chain that we will call Chain A. Chain A has already established a presence in the market area with 29 stores currently operating. The main competitor, Chain B, presently has 15 stores open. For the sake of the application, we assume that Chain A is interested in opening a maximum of three additional stores within the metropolitan area. Since Chain A has proposed 10 potential locations, we will use these to determine the combination of sites to optimize revenue.

We must first divide the metropolitan area into $N$ discrete demand points. Given the size of the metropolitan area and the vast amount of customer data under consideration, it is computationally impractical to consider each individual residence as a demand point. A common practice in location modeling is to aggregate demand points and then use the condensed data set to solve the problem [5]. In our case, we choose to consider the 117 postal zipcodes present within the metropolitan area and then use the centroid of each zipcode as demand point $i$.

Next, we calculate the expected utility for each of the proposed locations. Recall that $u_{ij} = \frac{A^\alpha_j}{d^\beta_{ij}}$. To simplify the application we let $A^\alpha_j = 1$, thus basing the utility mainly on how far customer $i$ is from store $j$. To do this, we first obtain distances from each existing and proposed location to each of the zipcode centroids. These distances are then weighted with the demand sensitivity parameter $\beta$. We will examine three levels of $\beta$ to determine what effect it has on our model and the solution generated.

We will also set the demand elasticity parameter $\lambda$ at three different levels, to compare the effects of low elasticity, moderate elasticity and high elasticity on the demand function $g_i(U)$. The market potential for each zipcode is based on a formula
developed by Chain A and we incorporate this information as \( w_i \) in the model \( M_0 \). The Multi-Facility Retail Location model for Chain A can now be formulated.

5.1 Optimization Model for Chain A

We can format the following optimization model for chain A using the previously developed model \( M_0 \).

\[
\begin{align*}
\text{max} & \quad \sum_{i=1}^{117} w_i g_i \left( \sum_{j \in E} u_{ij} + \sum_{j \in P} u_{ij} x_j \right) \left( 1 - \frac{\sum_{j \in C} u_{ij}}{\sum_{j \in E} u_{ij} + \sum_{j \in P} u_{ij} x_{ij}} \right) \\
\text{s.t.} & \quad \sum_{j=1}^{10} x_j \leq 3, \\
& \quad x_j \in \{0, 1\}, \quad j \in P
\end{align*}
\]

Using SAS programming software we solve this model with an integer programming procedure based on the simplex method. We input the data using the sparse data format where only the nonzero coefficients of the problem are entered.

As mentioned earlier, we will examine the results of varying the values of \( \beta \) and \( \lambda \). Recall that \( \beta \) determines just how sensitive the model is to distance. When \( \beta \) is low, studies have shown that stores are likely to be centrally located, where competitors are located in close proximity. This is also the case when the demand elasticity, \( \lambda \) is high[1]. A good example of this type of situation occurs with "big box" retail stores, where companies build a small number of large facilities. High values of \( \beta \) indicate that distance is a major factor in the location of facilities and as a result the optimal solution will often spread out the location of facilities. This is often referred to as a "service network" solution and occurs with the location of ATM machines and gas stations [1]. Similarly, high values of \( \lambda \) will also result in a focus on local markets.

The table shows that there is a marked difference in the projected sales among
the different levels of $\beta$. When there is low sensitivity to distance, $\beta = 1$, store E is selected as part of the optimal solution and projected sales for all stores are cumulatively over a billion. A more conservative estimate of projected sales is seen when $\beta = 2.507$. This level of $\beta$ is based on the research of Ghosh and Craig and yields an optimal solution of 120 to 140 million. This is more in line with average sales for the metropolitan area under consideration. Interestingly, it appears that building sites A, D, and J will almost always be part of the optimal solution, regardless of the varying parameters. Since this is the case we choose the solution generated when $\beta = 2.507$ and $\lambda = .5$ as our optimal solution. Since $\lambda$ is unknown for the market area being considered, we feel that it is appropriate to assume that demand is moderately elastic.

Using the parameters mentioned above, we can now examine the solution generated for our integer program. Using the optimal solution generated by $M_\alpha$, we can determine the estimated revenue by substituting the solution into the objective function of the original model $M_0$. First, the revenue of the 29 existing stores before any new stores are built is estimated to be $1341.9$ million. After the three new stores are built the revenue increases to $1391.7$, an increase of $7.31\%$. The total difference in revenue then is $42.189$ million, while the total revenue generated by the new stores is $98.143$ million. This is a clear indication of the new stores cannibalize the existing stores.
Table 2: Revenue Summary

<table>
<thead>
<tr>
<th>Stores</th>
<th>Estimated Revenue</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Before New Stores</td>
<td>$1341.9</td>
<td></td>
</tr>
<tr>
<td>Total After New Stores</td>
<td>$1391.7</td>
<td>7.31% (increase)</td>
</tr>
<tr>
<td>Store A</td>
<td>$29.505</td>
<td>2.13% (of total)</td>
</tr>
<tr>
<td>Store D</td>
<td>$27.407</td>
<td>1.98% (of total)</td>
</tr>
<tr>
<td>Store J</td>
<td>$41.231</td>
<td>2.98% (of total)</td>
</tr>
</tbody>
</table>

Estimated Revenue given in millions

balizing some of the sales from the previously existing stores. While cannibalization is a concern, the new stores generate enough new revenue to more than compensate for this amount. Next we can look at each of the new stores individually and see how each affects the objective function. As previously noted, we see the optimal solution is to build Stores A, D and J. The sales generated by stores A, D, and J are $29.505, $27.407 and $41.231 million respectively. Clearly, it appears that all three stores will have a significant effect on the revenue generated by the company, with Store J perhaps having the most impact at 2.98%.

5.2 Comparison with an Analyst’s Optimal Solution

To determine just how effective our model is we now compare our results, displayed in table 2, to the optimal sites chosen by the market analyst. The analyst has over 12 years of experience with this particular market and has developed a detailed assessment of the ten proposed sites. The top sites in his opinion are locations D, E, I and J, with a tie between sites D and E. Of these stores, store E has been open for three months and is performing well. Recall, that store E was included in the optimal solution when \( \beta = 1 \). Thus, while our model did not find it optimal to build Store E when sites A, D and J were selected this does not mean that Store E would not be a profitable site. While not selected as part of the optimal solution, the
company may have chosen to build in this location in order to increase its market presence or to prevent future competitive encroachment in this area of the market. In regard to store D, the analyst notes that even though there is a sister store nearby, there is still plenty of room for expansion in this area of the market. According to the analyst, a second store would also perform well. The final store choice that our optimal solution shared with the analyst is store J. We note that this store had a major impact on the optimal solution with estimated sales of $41.231 million. This site is located in the heart of the retail market and will likely be pursued by Chain A.

The only major discrepancy between our optimal solution and the stores chosen by the analyst is store A. Instead of selecting site A, the analyst preferred site I. When questioned about this difference he stated that site I is a new retail area in a rapidly growing suburb while site A is located in a blue collar area with a major competitor located nearby. Apparently site A is located at the edge of the retail market and as a result the location of the competitor was not included in the data our solution was based on. While, it may be interesting to rerun the analysis with the correct data it is encouraging to note that on the whole our optimal solution agreed with the analyst’s recommendations.
In this paper we introduced a model for locating multiple retail facilities with concave demand. Our model not only captures the effect of market expansion, it also considers the cannibalization effects from the competition and the company’s own stores. We then approximated this model using the $\alpha$-optimal TLA procedure developed by Aboolian et al. We then generated an optimal solution for a metropolitan area with real market data using SAS linear programming software. Based on the agreement between our optimal solution and that of the analyst we conclude that our model is appropriate for the company model we have presented.

As with all optimization models, however, there is always room for improvement. For instance, the objective function could be adjusted to maximize profit rather than revenue. Further, adjustments could take into consideration the cost of building and operating new stores and then weigh these costs against revenue produced by each site. In certain areas the costs of operation may outweigh the profit or revenue produced by a new store. By incorporating such factors into the objective function, our model could then determine an equally viable alternative, albeit with less expensive operating costs. We also suggest surveying the markets under consideration in order to obtain appropriate values for $\beta$ and $\lambda$. This would allow us to calculate more precise sales estimates for each site.

In the future, we would like to include other estimators of store attractiveness, such as parking, store design, and ease of access. While, we feel that our results were quite successful based on distance alone, including these other measures may prove to be quite significant in the utility consumers assign to facilities. As retail location strategies continue to evolve, the need for location algorithms will continue. In the meantime, we hope that we have provided retailers with a valuable planning tool.
REFERENCES


A. SAS Code

PROC IMPORT OUT= WORK.stores
   DATAFILE= "K:\Thesis\Data\data for SAS1.xls"
   DBMS=EXCEL REPLACE;
      SHEET="Sheet1$";
      GETNAMES=YES;
      MIXED=NO;
      SCANTEXT=YES;
      USEDATE=YES;
      SCANTIME=YES;
RUN;
proc print data=work.stores;
run;

proc contents data=stores;
run;

data stores2;
set stores ;
w_i = mktphelty;
array store(54) L1--P54;
array u_i(54);
beta=2.357
  ;
do j=1 to 54;
u_i(j)=1/(store(j)+1)**beta;
end;
drop beta j mktphelty;
run;
data utility;
set stores2(keep=u_i45-u_i54 rename=(u_i45=u1 u_i46=u2 u_i47=u3 u_i48=u4 u_i49=u5 u_i50=u6 u_i51=u7 u_i52=u8 u_i53=u9 u_i54=u10));
run;

proc means data=stores2 sum;
var u_i1-u_i54;
ods output summary= U_i;
run;
quit;
data stores3;
set u_i(drop=vname_u_i1-vname_u_i54);
UE=sum(of u_i1_sum--u_i29_sum);
UC=sum(of u_i30_sum--u_i44_sum);
UA=sum(UE, UC);
phibar=1.4210738 /*2.299317084*/;
run;
PROC IMPORT OUT= WORK.ziplam
   DATAFILE= "K:\Thesis\Data\zipcode and lambda.xls"
   DBMS=EXCEL REPLACE;
   SHEET="Sheet3$";
   GETNAMES=YES;
MIXED=NO;
SCANTEXT=YES;
USEDATE=YES;
SCANTIME=YES;
run;
PROC IMPORT OUT= WORK.storesFinal
   DATAFILE= "K:\Thesis\Data\stores5.xls"
   DBMS=EXCEL REPLACE;
   SHEET="Sheet1$";
   GETNAMES=YES;
   MIXED=NO;
   SCANTEXT=YES;
   USEDATE=YES;
   SCANTIME=YES;
run;
proc iml;
use storesfinal;
read all into z;
print z;
rows=nrow(z);
cols=ncol(z);
/*cols are zipcodes, w_i, lambda_i, UC, UE, UA*/
w=J(rows,12,0);
c=J(rows,13,0);/*define a matrix to put the c values*/
b=J(rows,12,0);/*define a matrix to put the b values*/
a=J(rows,12,0);/*define a matrix to put the a values*/
e=J(rows,12,0);/*define a matrix to put the ce values*/
LiUbar=J(rows,1,0); /* define a matrix to put the LiUbar values */
start fun1;
    f1=(w[j,8]+b[j,k]*(x-c[j,1])=((UE+x)/(UA+x))
        *(z[j,2]*(1-exp(-z[j,3]*(UA+x)))*(1+alpha));
/* evaluate the function */
finish fun1;
start deriv1;
    h1=(w[j,10]-(z[j,2]*(((z[j,6]-z[j,5])/(z[j,6]+x)**2)*
            (w[j,1])+z[j,3]*(z[j,5]+x)
            /(z[j,6]+x)*exp(-z[j,3]*(z[j,6]+x))));
finish deriv1;
start fun2;
    Wx = z[j,2]*(1-exp(-z[j,3]*(UA+x)))*((UE+x)/(UA+x));
    dWx=z[j,2]*(((Udif)/(UA+x)**2)*1-exp(-z[j,3]*(UA+x))
        +z[j,3]*(UE+x)/(UA+x)*exp(-z[j,3]*(UA+x)));
    f2=dWx*(x-c[j,1])+d-Wx;
/* evaluate the function */
finish fun2;
start deriv2;
    h2=z[j,2]*(-2*Udif/(UA+x)**3)*(1-exp(-z[j,3]*(UA+x)))
        +(2*Udif/(UA+x)**2)*exp(-z[j,3]*(UA+x))-z[j,3]
        *(exp(-z[j,3]*(UA+x)))/(UA+x);
finish deriv2;
start newton1;
    run fun1; /* evaluate function at starting values */
    do iter=1 to maxiter /* iterate until maxiter iterations */
        while(max(abs(f1))>converge); /* or convergence */
run deriv1; /* evaluate derivatives in j */
delta=-solve(h1,f1); /* solve for correction vector */
x=x+delta; /* the new approximation */
run fun1; /* evaluate the function */
end;
finish newton1;

start newton2;
run fun2; /* evaluate function at starting values */
   do iter=1 to maxiter /* iterate until maxiter iterations */
      while(max(abs(f2))>converge); /* or convergence */
         run deriv2; /* evaluate derivatives in j */
         delta=-solve(h2,f2); /* solve for correction vector */
         x=x+delta; /* the new approximation */
         run fun2; /* evaluate the function */
   end;
finish newton2;
maxiter=15; /* default maximum iterations */
converge=.000000001; /* default convergence criterion */

   do j=1 to rows;
      phi=0;
      UA=z[1,6];
      UE=z[1,5];
      Udif=UA-UE;
      w[j,1]=1-exp(-z[j,3]*(UA+0));
      w[j,2]=(z[j,2]*w[j,1])*((z[j,5]+phi)/(z[j,6]+phi));
      w[j,3]=z[j,2]*(((z[j,6]-z[j,5])/(z[j,6]+0)**2)*(w[j,1])
       +z[j,3]*(z[j,5]+0)/(z[j,6]+0)*exp(-z[j,3]*(z[j,6]+0)));

\[ w[j, 4] = 1.4210738 \]

\[ \text{phibar} = w[j, 4]; \]

\[ w[j, 5] = z[j, 2] * (1 - \exp(-z[j, 3] * (z[j, 6] + w[j, 4]))) \]
\[ * ((z[j, 5] + w[j, 4]) / (z[j, 6] + w[j, 4])); \]

\[ \text{Wphibar} = w[j, 5]; \]

\[ w[j, 6] = z[j, 2] * \left( (\text{UA} - \text{UE}) / (\text{UA} + \text{phibar})^{*2} \right) * (1 - \exp(-z[j, 3] * (\text{UA} + \text{phibar}))) \]
\[ + z[j, 3] * (\text{UE} + \text{phibar}) / (\text{UA} + \text{phibar}) \]
\[ * \exp(-z[j, 3] * (\text{UA} + \text{phibar})); \]

\[ d\text{Wphibar} = w[j, 6]; \]

\[ l = 1; \]

\[ k = 1; \]

\[ c[j, 1] = 0; \]

\[ b[j, 1] = w[j, 3]; \]

\[ \text{alpha} = 0.0001; \]

\[ \text{stop} = 0; \]

\[ \text{do until (stop = 1)}; \]

\[ w[j, 7] = 1 - \exp(-z[j, 3] * (\text{UA} + c[j, 1])); \]

\[ w[j, 8] = (z[j, 2] * w[j, 7]) * ((\text{UE} + c[j, 1]) / (\text{UA} + c[j, 1])); \]

\[ w[j, 9] = w[j, 5] + b[j, k] * (\text{phibar} - c[j, 1]); / \]

\[ w[j, 10] = z[j, 2] \]
\[ * ((z[j, 6] - z[j, 5]) / (z[j, 6] + c[j, 1])^{*2}) \]
\[ * (w[j, 1]) + z[j, 3] * (z[j, 5] + c[j, 1]) / (z[j, 6] + c[j, 1]) \]
\[ * \exp(-z[j, 3] * (z[j, 6] + c[j, 1])); \]

\[ \text{Lphibar} = w[j, 9]; \]

\[ \text{if Lphibar < Wphibar*(1+alpha)} \]

\[ \text{then do}; \]

\[ l = l + 1; \]

\[ c[j, 1] = w[j, 4]; / * \text{phibar} */ \]
\[ a[j,l-1] = c[j,l] - c[j,l-1]; \]
\[ LiUbar[j,1] = l; \]
\[ stop = 1; \]
\[ end; \]

else do;

\[ x = (c[j,l] + 0.00001); \] /* starting value */
\[ run newton1; \]
\[ print x j; \]
\[ e[j,1] = x; \]
\[ ce = e[j,1]; \]
\[ l = l + 1; \]
\[ c[j,1] = e[j,1-1]; \]
\[ a[j,l-1] = c[j,l] - c[j,l-1]; \]
\[ LiUbar[j,1] = l; \]
\[ if ce >= 1.4210738 \]
\[ /* 2.299317084*/ \]
\[ then do; \]
\[ c[j,1] = phibar; \]
\[ stop = 1; \]
\[ end; \]

else do;

\[ w[j,11] = z[j,2] * (1 - \exp(-z[j,3]*(UA+cE)))*((UE+ce)/(UA+ce))*(1+alpha); \]
\[ d = w[j,11]; \]
\[ if Wphibar < dWphibar*(phibar-ce)+d \]
\[ then do; \]
\[ l = l + 1; \]
\[ k = k + 1; \]
c[j,1]=phibar;
a[j,l-1]=c[j,1]-c[j,l-1];
LiUbar[j,1]=l;
if (z[j,2]*(1-exp(-z[j,3]*(UA+c[j,l-1]))))*(1+alpha) > Wphibar
then do;
b[j,k]=0;
stop=1;
end;
else do;
b[j,k]=(Wphibar-((1+alpha)*(z[j,2]*
(1-exp(-z[j,3]*(UA+c[j,1])))**(1+alpha) > Wphibar
then do;
b[j,k]=0;
stop=1;
end;
else do;
b[j,k]=(Wphibar-((1+alpha)*(z[j,2]*
(1-exp(-z[j,3]*(UA+c[j,1])))**(1+alpha) > Wphibar
then do;
b[j,k]=0;
stop=1;
end;
else do;
x=c[j,1]+.00001;
run newton2;
e[j,2]=x;
cT= e[j,2];
k=k+1;
b[j,k]=z[j,2]*(((Udif)/(UA+cT)**2)*(1-exp(-z[j,3]*(UA+cT)))
+ z[j,3]*(UE+cT)/(UA+cT)*exp(-z[j,3]*(UA+cT)))
if cT >=1.4210738
then do;
l=l+1;
c[j,1]=cT;
a[j,l-1]=c[j,1]-c[j,l-1];
LiUbar[j,1]=1;
stop=1;
end;

end;
end;
end;
end;
end;

end;
print a b;
final=J(rows,14,0);
do j=1 to rows;
final[j,1]=z[j,1];
final[j,2]=z[j,2];
do m=3 to 14;
final[j,m]=a[j,m-2];
end;
end;
CREATE lpdata FROM final;
APPEND FROM final;
zipb=J(rows,14,0);
do j=1 to rows;
zipb[j,1]=z[j,1];
zipb[j,2]=LiUbar[j,1];
do m=3 to 14;
zipb[j,m]=b[j,m-2];
end;
end;
create lpdata2 from zipb;
append from zipb;
quit;
data lpdata_;  
set lpdata(rename=(col1=zipcode col2=wi col3=a1 col4=a2 col5=a3
col6=a4 col7=a5 col8=a6 col9=a7 col10=a8 col11=a9 col12=a10
col13=a11 col14=a12));
run;
data lpdata2_;  
set lpdata2(rename=(col1=zipcode col2=LiUbar col3=b1 col4=b2 col5=b3
col6=b4 col7=b5 col8=b6 col9=b7 col10=b8 col11=b9 col12=b10
col13=b11 col14=b12));
run;
data final2;
merge lpdata_ lpdata2_ ;
by zipcode;
run;
data test1;
set final2;
array a(12);
array b(12);
do j=1 to 12;
_type_='max';
_row='obj';
_col_=cats('y_',_n_,'_',j);
_coef=wi*a(j)*b(j);
if _coef ne 0 then output;
end;
keep _type_ _row _col_ _coef ;
run;
data test2;
array x(10);
do j=1 to 10;
_type_='EQ';
_row='rw2';
_col_=cats('x_',j);
_coef=1;
if _coef ne 0 then output;
end;
_col_=cats('_rhs_');
_coef=3;
if _coef ne 0 then output;
keep _type_ _row _col_ _coef ;
run;
data test3;
set final2;
array a(12);
_type_='EQ';
do j=1 to 12;
_row=cats('rwa',_n_);
_col_=cats('y',_n_,',',j);
_coef= -a(j);
if _coef ne 0 then output;
end;
keep _type_ _row _col_ _coef;
run;
data test4;
set utility;
array u(10);
_type_='EQ';
do j=1 to 10;
_col_=cats('x',j);
_coef= u(j);
_row=cats('rwa',_n_);
if _coef ne 0 then output;
end;
keep _type_ _row _col_ _coef;
run;
data test5;
format _type_ $8. _col_ $14. _row $16.;
do j=1 to 117;
_type_='EQ';
_row=cats('rwa',j);
_col_=cats('_rhs_');
_coef= 0;

if _coef = 0 then output;
end;
keep _type_ _row _col_ _coef;
run;
data test6;
_type_='binary';
_row='binary';
do j=1 to 10;
_col_=cats('x_',j);
_coef= j;
if _coef ne 0 then output;
end;
keep _type_ _row _col_ _coef;
run;
data test7;
set final2;
do j=1 to 12;
_type_='LE';
_coef= 1;
_row=cats('rwb',_n_,'_',j);
_col_=cats('y_',_n_,'_',j);
if _coef ne 0 then output;
end;
keep _type_ _row _col_ _coef;
run;
data test8;
set final2;
do j=1 to 12;
_type_='LE';
_coef= 1;
_row=cats('rwb',_n_,'_',j);
_col_=cats('_rhs_');
if _coef ne 0 then output;
end;
keep _type_ _row _col_ _coef;
run;

proc append base=test data=test1;
run;
proc append base=test data=test2 force;
run;
proc append base=test data=test3 force;
run;
proc append base=test data=test4 force;
run;
proc append base=test data=test5 force;
run;
proc append base=test data=test6 force;
run;
proc append base=test data=test7 force;
run;

proc append base=test data=test8 force;
run;

PROC LP data=test sparsedata maxit1=1000 maxit2=1000;
RUN;