TWO-PATCH PREDATOR-PREY SYSTEM COUPLED WITH MIGRATION
OF BOTH SPECIES

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ABSTRACT

In this paper we explore the dynamics of predator-prey in a two patch system. The two patches of the system are coupled with both the migration of the predator and the prey. The purpose of this exploration is to find upper and lower bounds for the populations and get an insight on the different possibilities with the three types of Holling functional responses. Also we discuss the stability and instability of the equilibrium solutions found in earlier papers. Numerical simulations are provided to graphically demonstrate the population dynamics of the system.
DEDICATION

I dedicate this thesis to my mom and dad for presenting me with the opportunity to attend the University of North Carolina Wilmington. To my family and friends, thank you for your continued encouragement and support.
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I would like to take this time to show my appreciation and gratitude to those who have helped me along the way. Dr. Feng: You are my mentor. Throughout my college career, you have been the one to push me the hardest. Your consistent belief in my ability to be a better student - apparent even during our most heated debates - has provided me with the strength to excel in my studies. I would also like to thank my committee members, Dr. lu and Dr. Hou for their help along the way.

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1 HISTORY

Perhaps one of the simplest, oldest growth models was the Malthusian growth model. Named after the Reverend Thomas Malthus, the model was represented by

\[ P(t) = P_0 e^{rt}, \]  

where \( P_0 \) = Initial Population, \( r \) = growth rate, sometimes also called Malthusian Parameter, and \( t \) = time [19]. Next to come along was the logistic equation,

\[ \frac{dP}{dt} = rP \left( 1 - \frac{P}{K} \right) \]  

by Pierre-Francois Verhulst [19]. The logistic equation represents a one population system where \( r \) is the intrinsic birth rate and \( K \) is the carrying capacity. In this model the population just competes with each other over the food resources. The next model is the Lokta-Volterra model and is the basis for many two species models. The model is represented by,

\[ \frac{dx}{dt} = x(\alpha - \beta y) \]
\[ \frac{dy}{dt} = -y(\gamma - \delta x) \]  

where \( x \) is the prey \( y \) is the predator. The density functions \( \frac{dy}{dt} \) and \( \frac{dx}{dt} \) represents the growth of the two populations against time, \( t \). The parameters \( \alpha \) and \( \gamma \) are considered the intrinsic growth rates. Where as the other parameters \( \delta \) and \( \beta \) are representing the interaction of the two species [19]. The basic Lokta-Volterra model describes dynamics of biological systems in which two species interact, one a predator and one its prey.
This paper was influenced by an article that was published by Jody Hinson and Wei Feng in 2005. The article done by Hinson and Feng was focused on a model developed by V.A.A. Jansen.

\[
\frac{dN_i}{dt} = rN_i \left(1 - \frac{N_i}{K}\right) - \frac{bN_iP_i}{b + N_i}
\]

\[
\frac{dP_i}{dt} = \frac{bN_iP_i}{b + N_i} - \mu P_i + d \left(\frac{P_j}{1 + N_j} - \frac{P_i}{1 + N_i}\right)
\]

\[i, j \in \{1, 2\} : i \neq j.\]

Where the variables \(N_i\) and \(P_i\) denote the densities of the prey and predator, respectively, in patch \(i\). The parameter \(d\) denotes the per capita predator migration rate. The parameter \(r\) is the prey growth rate at low prey densities; \(K\) is the environmental carrying capacity of the prey population. The parameter \(\mu\) is the predator death rate in the absence of prey, and \(b\) is the saturation value of the functional response [13, 12, 11, 10]. The stability results in Hinson’s thesis were based on seven of the twelve equilibrium existing in the model. Hinson then ran numerical simulations on the model to demonstrate asymptotic behavior over time.

This thesis is based on a slightly more complex model,

\[
\frac{dN_i}{dt} = rN_i \left(1 - \frac{N_i}{K}\right) - \frac{bN_iP_i}{b + N_i} + m(N_j - N_i)
\]

\[
\frac{dP_i}{dt} = \frac{bN_iP_i}{b + N_i} - \mu P_i + d \left(\frac{P_j}{1 + N_j} - \frac{P_i}{1 + N_i}\right)
\]

\[i, j \in \{1, 2\} : i \neq j.\]
The parameters of this model and the previous are the same, except that \( m \) is migration of prey. This model is constructed to involve both the migration of the predator and the prey. Where as the previous model by Jansen only had migration of the predator. Another difference in the model can be seen in the migration of the predator. This model allows for migration of the predator to be dependent on the population of the prey in each patch.

The purpose of this paper is to analyze and demonstrate the population dynamics of the various species in model number by

1. finding ultimate upper and lower bounds for the interacting populations;

2. exploring the stability and instability of the equilibrium solutions;

\[
E_1 = (0, 0, 0, 0)
\]

\[
E_2 = (K, 0, K, 0)
\]

\[
E_3 = \left( \frac{b \mu}{b - \mu}, \left( \frac{rb}{b - \mu} \right) \left( 1 - \frac{b \mu}{K(b - \mu)} \right), \frac{b \mu}{b - \mu}, \left( \frac{rb}{b - \mu} \right) \left( 1 - \frac{b \mu}{K(b - \mu)} \right) \right)
\]

3. obtaining numerical simulations for the pattern of dynamics in the model.
The two-patch predator-prey model is given by,

\[
\frac{dN_i}{dt} = rN_i \left(1 - \frac{N_i}{K}\right) - \frac{bN_iP_i}{b + N_i} + m(N_j - N_i)
\]

\[
\frac{dP_i}{dt} = \frac{bN_iP_i}{b + N_i} - \mu P_i + d \left(\frac{P_j}{1 + N_j} - \frac{P_i}{1 + N_j}\right)
\]

(4) \hspace{1cm} i, j \in \{1, 2\} : i \neq j.

Where \(N_i\) and \(P_i\) denote the densities of the prey and predator, respectively, in patch \(i\). The parameter \(d\) denotes the per capita predator migration rate and \(m\) is the migration rate of the prey. The parameter \(r\) is the prey growth rate at low prey densities; \(K\) is the environmental carrying capacity of the prey population. The parameter \(\mu\) is the predator death rate in the absence of prey, and \(b\) is the saturation value of the functional response.

First looking at the differential equations of the prey we can determine what parts of the equation affect the fluctuation of the population. The first term in the prey equation, \(rN_i \left(1 - \frac{N_i}{K}\right)\) comes from the simple logistic equation. Recall the logistic equation just models the competition for resources among that species. We then see that the second term \(-\frac{bN_iP_i}{b + N_i}\), is depleting the prey population. This is the Holling type II disc equation. It is designed to represent consumption of the prey by the predator. The last term \(m(N_j - N_i)\) represents the migration of the prey from one patch to another. The migration of the prey is dependent on the migration constant \(m\) and the number of prey in each patch.

The first term in the density equation for the predator is \(\frac{bN_iP_i}{b + N_i}\). This shows an increase in the predator population due to the consumption of the prey. One then
can see a decrease in the population by $\mu P_i$. This is the natural death rate of the predator, since our predators have no threats in this model. The main focus in the predator equation is in the term involving the migration, $d \left( \frac{P_i}{1+N_j} - \frac{P_j}{1+N_i} \right)$. The interesting attribute to notice here is that the predators migrate according to the size of the prey in each patch. We will now look at an numerical example of a single patch to better illustrate how the predator’s migrate.

Example 1. Recall,

$$\frac{dP_1}{dt} = \frac{bN_1P_1}{b+N_1} - \mu P_1 + d \left( \frac{P_2}{1+N_2} - \frac{P_1}{1+N_1} \right)$$

This is the density equation for the predator population in patch $P_1$. By selection a set of parameters for the prey and predators, we can see how the predator population will migrate depending on the number of prey. By choosing the values $N_2 = 20$, $N_1 = 10$, $P_2 = 10$, and $P_1 = 10$ this will illustrate an equal number of predators in each patch and more prey in patch two. One would hope that the migration term for the predator population would be negative, illustrating the predator migrating to the patch with more prey. Focusing on the migration parameter we see,

$$d \left( \frac{10}{1+20} - \frac{10}{1+10} \right)$$

$$= d (.47 - .90) .$$

Thus this gives a negative value for the migration of the predators. Reflecting the assumption that the predator will migrate to the patch that has the most prey.
Typically in the study of mathematical models, one would like to be assured that the model realistically represents the real-world situation. That is one needs to verify that the populations of the predator and prey are non-negative and within reasonable ultimate bounds. Before we find the ultimate bounds for the populations, we need to recall two lemma’s that provide the basic tools for qualitative analysis of the solution [1, 6, 5].

**Lemma 1.** *Positivity Lemma.* Let $W(t)$ be a smooth function in $[0, T]$. If $W(t)$ satisfies $W'(t) + M(t)W(t) \geq 0$ in $(0, T]$ and $W(0) \geq 0$, where $M(t)$ is a bounded function in $[0, T]$, then $W(t) \geq 0$ on $[0, T]$.

**Proof.** This lemma is proved by contradiction. Assume that the statement $W(t) \geq 0$ in $[0, T]$ is false, then there exists a point $t_0 \in [0, T]$ such that $W(t_0)$ is a negative minimum of $W(t)$ on $[0, T]$. Since $W(0) \geq 0$, then $t_0 \in (0, T]$ which means that $W'(t_0) + M(t_0)W(t_0) \geq 0$. Since $W(t)$ reaches its minimum value at $t_0$, then $W''(t_0) = 0$ if $t_0 \neq T$ and $W'(t_0) \leq 0$ if $t_0 = T$. This ensures that $M(t_0)W(t_0) \geq 0$ which contradicts the assumption about $W(t_0) < 0$ when $M(t_0) > 0$.

For the case of $M(t_0) \leq 0$, we let $V(t) = e^{-\sigma t}W(t)$ for some constant $\sigma$ with $\sigma > -M(t)$ in $(0, T]$, then $V(t)$ will satisfy the relation $V''(t) + (\sigma + M(t))V(t) \geq 0$ in $(0, T]$ and $V(0) \geq 0$, where $\sigma + M(t) > 0$ for all $t \in (0, T]$. From the above arguments we have $V(t) \geq 0$ in $[0, T]$. It follows from $W(t) = e^{\sigma t}V(t)$ that $W(t) \geq 0$ on $[0, T]$.

As an application of Lemma 1, I have the following comparison argument [5, 15, 18] for the respective solutions $u_1$ and $u_2$ of the initial-value problem

$$u_i' = f_i(t, u_i) \text{ in } (0, T], \quad u_i(0) = u_{i,0} \quad (5)$$
where \( i = 1, 2 \). \( f_1 \) and \( f_2 \) are continuous functions in \([0, T] \times R\).

**Lemma 2.** The Comparison Argument. Assume that both \( \frac{\partial f_1}{\partial u} \) and \( \frac{\partial f_2}{\partial u} \) are continuous in \([0, T] \times R\). If \( f_1(t, u) \leq f_2(t, u) \) in \((0, T] \times R\) and \( u_{1,0} \leq u_{2,0} \), then the respective solutions \( u_1 \) and \( u_2 \) of (5) satisfy \( u_1(t) \leq u_2(t) \) on \([0, T]\).

**Proof.** Let \( W(t) = u_2(t) - u_1(t) \), and let \( M(t) \) be any bounded function in \([0, T]\).

Then by Lemma 1, \( W(t) \) satisfies

\[
W'(t) + M(t)W(t) = M(t)[u_2(t) - u_1(t)] + f_2(t, u_2(t)) - f_1(t, u_1(t)) \text{ in } (0, T] \\
W(0) = u_{2,0} - u_{1,0} \geq 0.
\]

Since \( \frac{\partial f_1}{\partial u} \) is continuous in \( u \), then by the mean value theorem,

\[
f_2(t, u_2) - f_1(t, u_1) = [f_2(t, u_2) - f_1(t, u_2)] + [f_1(t, u_2) - f_1(t, u_1)] \\
\geq \frac{\partial f_1}{\partial u}(t, \tilde{\eta})(u_2 - u_1)
\]

where \( \tilde{\eta} = \tilde{\eta}(t) \) is an intermediate value between \( u_1 \) and \( u_2 \). Hence, for the bounded function \( M(t) = -\frac{\partial f_1}{\partial u}(t, \tilde{\eta}) \), \( W(t) \) satisfies \( W'(t) + M(t)W(t) \geq 0 \) in \((0, T]\). It is known from Lemma 1 that \( W(t) \geq 0 \), i.e. \( u_2(t) \geq u_1(t) \) on \([0, T]\). This proves Lemma 2.

**4.1 Ultimate Bounds for the Prey Population**

In this section we plan to verify the upper and lower bounds on the prey populations. We also seek for conditions that the upper bounds of the prey population do not shot to infinity or tend towards zero. Finding these conditions is crucial because it
provides us with information on extinction, co-existence, and exponential behavior of the species.

**Theorem 1.** **Ultimate bounds for prey populations**

\[
0 \leq N(t) \leq \left[ \left( \frac{1}{N(0)} - \frac{1}{2K} \right) e^{-rt} + \frac{1}{2K} \right]^{-1} \leq 2K,
\]

where \( N(t) = N_1(t) + N_2(t) \).

**Proof.** Recall in (4) that the original prey population,

\[
\frac{dN_1}{dt} = rN_1 \left( 1 - \frac{N_1}{K} \right) - \frac{bN_1P_1}{b + N_1} + m(N_2 - N_1).
\]

The non-negativity of the density functions allows one to discard some terms leaving,

\[
\frac{dN_1}{dt} \leq rN_1 \left( 1 - \frac{N_1}{K} \right) + m(N_2 - N_1),
\]

\[
\frac{dN_2}{dt} \leq rN_2 \left( 1 - \frac{N_2}{K} \right) + m(N_1 - N_2).
\]

We now seek to add the prey populations in both patches to obtain the ultimate upper bound of the system. After adding the two inequalities we obtain,

\[
\frac{d}{dt}(N_1 + N_2) \leq rN_1 + rN_2 - \left( \frac{rN_1^2}{K} + \frac{rN_2^2}{K} \right).
\]

Now we can use the fact that,

\[
\frac{(N_1 + N_2)^2}{2} \leq N_1^2 + N_2^2.
\]

Thus we now see the inequality changes to,

\[
\frac{d}{dt}(N_1 + N_2) \leq rN_1 + rN_2 - \left( \frac{r(N_1 + N_2)^2}{2K} \right).
\]
Now letting $N = N_1 + N_2$ we obtain our final inequality,

$$\frac{dN}{dt} \leq rN - \left( \frac{rN^2}{2K} \right)$$

$$\frac{dN}{dt} \leq rN \left( 1 - \frac{N}{2K} \right).$$

This is now a solvable differential equation and can be recognized as a Bernoulli differential equation\(^1\). After solving the Bernoulli type two differential equation we obtain,

$$N(t) \leq \left[ \frac{1}{2K} + \frac{cr}{2Ke^{rt}} \right]^{-1} \leq \left[ \left( \frac{2K - N(0)}{N(0)r} \right) \frac{r}{2Ke^{rt}} + \frac{1}{2K} \right]^{-1}$$

We now conclude that the upper bounds on the total prey population is less then $2K$. Since,

$$\lim_{t \to \infty} N(t) \leq \lim_{t \to \infty} (N_1(t) + N_2(t)) \leq \lim_{t \to \infty} \left[ \left( \frac{1}{N(0)} - \frac{1}{2K} \right) e^{-rt} + \frac{1}{2K} \right]^{-1} = 2K$$

where $N(0) \neq 0$.

One would expect that the total of the prey populations can not exceed $2K$. If each patch has a certain carrying capacity $K$, that means that capacity cannot be breached. Having two patches that’s why we obtain the result of $2K$. The lower bound on the prey populations comes from non-negativity of the populations, hence the lower bound is zero.

\(^1\)A differential equation of the form $y' + p(x)y = q(x)y^n$ is called a Bernoulli differential equation. To solve the equation you first need to divide by $y^n$ to obtain, $y^{-n} + p(x)y^{1-n} = q(x)$. Then by preforming the substitution of $v = y^{1-n}$, one reduces the differential equation to one that can be solved using the method of integrating factor.
4.2 Ultimate Bounds for the Predator Population

We wish to apply the same techniques in finding the ultimate bounds for the predator populations as we did in finding the bounds for the prey populations. We should expect to see similar results in the lower bounds, but should notice some differences in the upper bounds.

**Theorem 2.** The upper bound for the total predator population is,

\[ 0 \leq P(t) \leq (P_1(0) + P_2(0))e^{\left(\frac{2bK}{2b+K} - \mu\right)t} \]

where, \( P(t) = P_1(t) + P_2(t) \).

**Proof.** Since

\[ \frac{dP_i}{dt} = \frac{bN_iP_i}{b+N_i} - \mu P_i + \frac{d}{P_j} \left( \frac{P_j}{1+N_j} - \frac{P_i}{1+N_i} \right) \]

where \( i = 1, 2 \). Then by adding the two predator patches we obtain,

\[ \frac{dP}{dt} = \frac{bN_1P_1}{b+N_1} - \mu P_1 + \frac{bN_2P_2}{b+N_2} - \mu P_2. \]

Now since \( N_1 \leq 2K \) and \( N_2 \leq 2K \) we have,

\[ \frac{dP}{dt} \leq \frac{2bKP_1}{b+2K} - \mu P_1 + \frac{2bKP_2}{b+2K} - \mu P_2. \]

Then by the comparison lemma this leads to the seperable differential inequality,

\[ \frac{dP}{dt} \leq \left( \frac{2bK}{b+2K} - \mu \right) P \]

Solving the differential equation and using the comparison argument in Lemma 2 leads to,

\[ P(t) \leq (P_1(0) + P_2(0))e^{\left(\frac{2bK}{2b+K} - \mu\right)t}. \]
This results leads us to the next corallary on the extinction of the predator populations.

**Corollary 1.** If $\frac{2bK}{b+2K} < \mu$, then both the predator populations are globally exponentially stable$^2$. If $\frac{2bK}{b+2K} = \mu$, then the predator population remains bounded.

This means that if $\frac{2bK}{b+2K} < \mu$, then both the predator populations go to extinction, $\lim_{t \to \infty} P(t) = 0$. Thus, one can see the survival of the predator is dependent on the carrying capacity of the prey, functional response, and the death rate. Notice that if $\frac{2bK}{b+2K} < \mu$, then the upper bound for the predator population will go to infinity. The idea here is that there is no restriction or carrying capacity on the predator population, thus they are not bounded above.

**Theorem 3.** Lower bounds for Predator populations

\[ P(t) \geq (P_1(0) + P_2(0))e^{-\mu t} \]

**Proof.** Recall,

\[ \frac{dP_i}{dt} = \frac{bN_iP_i}{b + N_i} - \mu P_i + d \left( \frac{P_j}{1 + N_j} - \frac{P_i}{1 + N_i} \right) \]

then,

\[ \frac{dP_1}{dt} \geq -\mu P_1 \]
\[ \frac{dP_2}{dt} \geq -\mu P_2. \]

Then by letting $P = P_1 + P_2$ we obtain,

\[ \frac{dP}{dt} \geq -\mu P \]

$^2$The origin is a globally exponentially stable equilibrium of $P'$ if there exists positive $\alpha$ and $\beta$, independent of $t_0$, such that $\|P(t)\| < \alpha \|P(t_0)\|e^{-\beta t}$ where $t \geq 0$. 

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This is a simple ordinary differential equation that is solved using separation of variables. After solving we get,

\[ P(t) \geq (P_1(0) + P_2(0))e^{-\mu t}, \]

Thus we see as \( t \) approaches infinity, then the exponential leads the ultimate lower bound for the predator population to zero. This happens since \( \mu \) is a positive parameters. Having the minimum of the predator population at zero directly reflects, a real life situation. It’s unrealistic to have a negative population.

Finding the ultimate bounds for the model is essential to the research. One has to first find that the bounds are realistic, before they begin doing analysis on the model. One would now like to find equilibrium solutions and determine stability.
5 STABILITY

Now that we have verified the ultimate bounds for the predator and prey populations, we can determine stability on the equilibrium solutions. In determining stability one would like to, first find the Jacobian\(^3\) of the model, determine the characteristic polynomial,\(^4\) and then find the roots of the characteristic polynomial. After one finds the roots of the polynomial, then stability is determined by whether or not all the roots are negative. If all the roots are negative then the local solutions around that equilibrium solution are stable. If one or more of the roots is positive then one will see the solutions around that equilibrium being pushed away. The following theorem lemma’s will help in determining the stability of the equilibrium solutions\([16, 1, 5]\).

Many problems in natural sciences involve rates of change dependent on the interaction of the elements. These problems are often expressed as a system of ordinary differential equations:

\[
y'_1 = f_1(y_1, \cdots, y_n) \quad \cdots \quad y'_n = f_n(y_1, \cdots, y_n). \quad (6)
\]

When the reaction functions \(f_i\) are linear in the variables \(y_i, i = 1, \cdots, n\), the system (6) takes the form

\[
y'_1 = a_{11}y_1 + \cdots + a_{1n}y_n \quad \cdots \quad y'_n = a_{n1}y_1 + \cdots + a_{nn}y_n. \quad (7)
\]

\(^3\)Named after the mathematician Carl Gustav Jacob Jacobi, the Jacobian matrix is the matrix of all first-order partial derivatives of a vector-valued function.

\(^4\)If there exist a field \(K\) (such as the real or complex numbers) and an square matrix \(A\) over \(K\). The characteristic polynomial of \(A\), denoted by \(p_A(t)\), is the polynomial defined by \(p_A(t) = \det(A - tI)\) where \(I\) denotes the square identity matrix and the determinant is being taken in \(K(t)\), the field of rational functions in \(t\). This is indeed a polynomial, since determinants are defined in terms of sums of products.
The linear system of ordinary differential equations (7) can be expressed as $Y' = AY$, where $A$ is an $n \times n$ constant matrix and $Y$ is a vector in $\mathbb{R}^n$. The trivial solution\(^5\) of (7) is said to be stable\(^13\) if for every $\epsilon > 0$, there is a $\delta > 0$ such that if $Y(t)$ is any solution of (7) with $\|Y(0)\| < \delta$, then $\|Y(t)\| < \epsilon$ for all $t > 0$. The trivial solution of (7) is said to be asymptotically stable\(^6\)[13] if (a) it is stable, and (b) there is a $r > 0$ such that if $\|Y(0)\| < r$, then $\lim_{t \to 0} \|Y(t)\| = 0$. The following lemma\(^13\) gives the stability of the trivial solution in (7):

**Lemma 3.** (i) The trivial solution of (7) is asymptotically stable if and only if all of the eigenvalues of $A$ have negative real parts.

(ii) If one eigenvalue of $A$ has a positive real part, then the trivial solution of (7) is unstable.

(iii) If the eigenvalues of $A$ with zero real parts are simple and all other eigenvalues have negative real parts, then the trivial solution of (7) is stable.

We now turn back to the general system (6). $y_0 = (y_0^1, \cdots, y_0^n) \in \mathbb{R}^n$ is called an equilibrium of (6) if $f_i(y_0^1, \cdots, y_0^n) = 0$ for $i = 1, \cdots, n$. The equilibrium $y_0$ for (6) is said to be stable if for every $\epsilon > 0$, there is a $\delta > 0$ such that if $Y(t)$ is any solution of (6) with $\|Y(0) - y_0\| < \delta$, then $\|Y(t) - y_0\| < \epsilon$ for all $t > 0$. The equilibrium $y_0$ of (6) is said to be asymptotically \([13, 1]\) stable if (a) it is stable, and (b) there is a $r > 0$ such that if $\|Y(0) - y_0\| < r$, then $\lim_{t \to \infty} \|Y(t) - y_0\| = 0$. Denote by $J$ the Jacobian matrix of $(f_1, \cdots, f_n)$ at $(y_0^1, \cdots, y_0^n)$. The next lemma [13] gives the relationship between the stability of trivial solution in linear system $X' = JX$ and the stability of the equilibrium $(y_0^1, \cdots, y_0^n)$ in (6):

\(^5\)Trivial solution refers to solutions to an equation that have a very simple structure, but for the sake of completeness cannot be omitted.

\(^6\)Asymptotically stable means that not only do initial conditions close to the origin stay close to the origin (stable), they also approach the origin asymptotically (the limit condition on the state). Therefore, ”asymptotic stability” is a stronger condition than plain ”stability” because it requires that trajectories satisfy more restrictive conditions. (this definition applies to the 2-d case)
Lemma 4. (i) If the trivial solution is asymptotically stable in the linear system $X' = JX$, then $(y_0^1, \ldots, y_0^n)$ is an asymptotically stable equilibrium for (6).

(ii) If the trivial solution is unstable in the linear system $X' = JX$, then $(y_0^1, \ldots, y_0^n)$ is an unstable equilibrium for (6).

5.1 Jacobian

Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ is a function from Euclidean $n$-space to Euclidean $m$-space. Such a function is given by $m$ real-valued component functions in (6). The partial derivatives of all these functions (if they exist) can be organized in an $m$-by-$n$ matrix, the Jacobian matrix $J$ of $F$, as follows:

\[
J = \begin{bmatrix}
\frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n}
\end{bmatrix}.
\]

The reason for using the Jacobian method for determining stability is because our model dealing with four differential equations. If we were dealing with a two dimensional system we would use a method of phase portraits\(^7\), to see how a system acts locally. The Jacobian for our model is represented by,

\[
J = \begin{bmatrix}
    r - \frac{2N_1}{K} - \frac{b^2P_1}{(b+N_1)^2} - m & -\frac{bN_1}{b+N_1} & m & 0 \\
    \frac{b^2P_1}{(b+N_1)^2} - \frac{dP_1}{(1+N_1)^2} & J_{22} & \frac{dP_2}{(1+N_2)^2} & \frac{d}{1+N_2} \\
    \frac{d}{1+N_2} & \frac{dP_2}{(1+N_2)^2} - \frac{b^2P_2}{(b+N_2)^2} - m - \frac{bN_2}{b+N_2} & J_{44}
\end{bmatrix}
\]

where, $J_{22} = \frac{bN_1}{b+N_1} - \mu - \frac{d}{1+N_1}$ and $J_{44} = \frac{bN_2}{b+N_2} - \mu - \frac{d}{1+N_1}$.

\(^7\)A phase portrait is a geometric representation of the trajectories of a dynamical system in the phase plane.
5.2 Stability of the trivial equilibrium solution

We now want to plug the equilibrium solution, $E_1$ into the Jacobian and find the characteristic polynomial. One would expect $E_1$ to be unstable, since this is the solution where there is no prey or predators in each of the patches. This is needed so that the solutions around this equilibrium are being pushed away, as shown in Figure 1, The 2-d phase plane shows the local points leaving the trivial solution and approaching another solution. If $E_1$ was attracting solutions this would mean that the local points would be drifting towards zero [19, 2, 9]. Hence, extinction of the species.

**Theorem 4.** The solution $E_1 = (0,0,0,0)$ is unstable.

*Proof.* After inserting the trivial equilibrium solution into the Jacobian we obtain,

$$J_{(0,0,0,0)} = \begin{bmatrix} r - m & 0 & m & 0 \\ 0 & -\mu - d & 0 & d \\ m & 0 & r - m & 0 \\ 0 & d & 0 & -\mu - d \end{bmatrix}.$$

Then taking the determinate of the Jacobian subtracted by the identity matrix $\lambda I$ leaves,

$$\det |J - \lambda I| = \begin{vmatrix} r - m - \lambda & 0 & m & 0 \\ 0 & -\mu - d - \lambda & 0 & d \\ m & 0 & r - m - \lambda & 0 \\ 0 & d & 0 & -\mu - d - \lambda \end{vmatrix} = 0.$$
We then obtain the characteristic polynomial of,

\[(λ + 2d + μ)(λ + μ)(λ^2 + (2m - 2r)λ + r^2 - 2rm).\]

Thus we see that the roots of the Characteristic polynomial are $-2d - μ$, $-μ$, $r$, and $r - 2m$. Therefore we see that we have at least one positive root and this leads to the conclusion of instability.

The $E_1$ equilibrium solution represents a situation where there does not exist prey or predator in each patch. If we came to the conclusion of stability, then that means all of the solutions around $E_1$ would be leading to extinction. This is why one would like this to be unstable. This would lead to the solutions around $E_1$ being pushed towards some other solution, one that does not lead to extinction.

5.3 Stability if prey exists with no predators

**Theorem 5.** The $E_2 = (K, 0, K, 0)$ solution is

(i) asymptotically stable if $\frac{bK}{b+K} < μ$

(ii) globally exponentially stable if $N_{i,j} \leq 2K$, and $\frac{2bK}{b+2K} < μ$

(iii) unstable if $\frac{bK}{b+K} > μ$

(iv) stable if $\frac{bK}{b+K} = μ$.

**Proof.** After inserting the equilibrium solution into the Jacobian we obtain,

\[
J_{(K,0,K,0)} = \begin{bmatrix}
-r - m & \frac{-bK}{b+K} & m & 0 \\
0 & \frac{bK}{b+K} - μ - \frac{d}{1+K} & 0 & \frac{d}{1+K} \\
m & 0 & -r - m & -\frac{bK}{b+K} \\
0 & \frac{d}{1+K} & 0 & \frac{bK}{b+K} - μ - \frac{d}{1+K}
\end{bmatrix}
\]
Then taking the determinate of the Jacobian subtracted by the identity matrix leaves,

\[
\det |J - \lambda I| = \begin{vmatrix}
  -r - m - \lambda & \frac{-bK}{b+K} & \mu & 0 \\
  0 & \zeta - \lambda & 0 & \frac{d}{1+K} \\
  \mu & 0 & -r - m - \lambda & \frac{-bK}{b+K} \\
  0 & \frac{d}{1+K} & 0 & \zeta - \lambda
\end{vmatrix} = 0
\]

where, \(\zeta = \frac{bK}{b+K} - \mu - \frac{d}{1+K}\). We then obtain the characteristic polynomial of,

\[-\frac{1}{(1+K)(b+K)^2}f_1(\lambda)f_2(\lambda)f_3(\lambda)f_4(\lambda) = 0\]

where \(f_n\) is just the factors of the characteristic polynomial. Thus,

\[f_1(\lambda) = (-b\lambda - K\lambda + bK - b\mu - K\mu)\]
\[f_2(\lambda) = (b(-K^2 + K\mu + K\lambda - K + 2d + \lambda + \mu) + K^2(\mu + \lambda) + K(2d + \lambda + \mu))\]
\[f_3(\lambda) = (r + \lambda)\]
\[f_4(\lambda) = (r + 2m + \lambda)\]

Then see that the roots of the characteristic polynomial are,

\[\lambda_1 = \frac{-bK + b\mu + K\mu}{-(b + K)}\]
\[\lambda_2 = \frac{bK^2 + bK - 2db - 2dK - \mu(bK + b + K^2 + K)}{(bK + b + K^2 + K)}\]
\[\lambda_3 = -r\]
\[\lambda_4 = -2m - r\]

Clearly, one can see that the \(\lambda_3\) and the \(\lambda_4\) roots are negative, since all the parameters
in the model are non-negative. Through a little algebra we are able to put some slight restrictions on the parameters to ensure that the remaining roots stay negative. The other two roots give the restrictions, \( \frac{bK}{b+K} < \mu \) and \( \frac{bK^2 + bK - 2db - 2dK}{(bK + b + K^2 + K)} < \mu \). Taking a closer look at \( \frac{bK^2 + bK - 2db - 2dK}{(bK + b + K^2 + K)} \) we see, 

\[
\frac{bK^2 + bK - 2db - 2dK}{(K + 1)(K + b)} < \mu.
\]

This leads to,

\[
bK - 2d + \frac{2d - 2db}{K + 1} < (b + K)\mu.
\]

Since we know \( \frac{bK}{b+K} < \mu \), we seek to show

\[
bK - 2d + \frac{2d - 2db}{K + 1} \leq bK < (b + K)\mu.
\]

This holds since \(-2d + \frac{2d - 2db}{K + 1} < 0\). Thus, we can conclude that the only stability condition for the \( E_2 \) solution is \( \frac{bK}{b+K} < \mu \). This proves (i).

From Corollary 1 we derived a global condition on the stability of the predators, by using the upper bound attained from the prey population, \( N_{i,j} \leq 2K \). While trying to find an upper bound we noticed that if \( \frac{2bK}{b+2K} < \mu \), then this would lead to the extinction of the predator. This proves (ii).

It will suffice just to show that one of the eigenvalues is positive for proving (iii). One can see that \( \lambda_3 \) and \( \lambda_4 \) are not positive, since we have all negative parameters. Thus, the conditions for instability lie in \( \lambda_1 \) and \( \lambda_2 \). One can further see that this is just the opposite for the condition of stability, i.e. \( \frac{bK}{b+K} < \mu \). As well one can see that we need one conditions to equal zero for stability, thus if \( \lambda_1 = 0 \), we have stability.
For an equilibrium to be stable all, one needs to show is that one of the roots equals zero. This we see that if we set $\lambda_1$ equal to zero we will obtain a sufficient condition for stability. Therefore we then see if $\frac{bk}{b+K} = \mu$, the $E_2$ is stable, proving (iv)

5.4 Stability with coexistence

We now study conditions for stability of $E_3$, which will ensure the long-term coexistence of the predator and prey.

**Theorem 6.** If $b > \mu$ and $\frac{bk}{b+K} > \mu$ then the system has a coexistent state at $E_3 = \left(\frac{b\mu}{b-\mu}, \frac{rb}{b-\mu}, \frac{b\mu}{b-\mu}, \frac{rb}{b-\mu}(1 - \frac{b\mu}{K(b-\mu)})\right)$. $E_3$ is asymptotically stable if $K < \frac{b^2 + \mu b}{b-\mu}$.

**Proof.** After inserting the equilibrium solution into the Jacobian we obtain,

$$J\left(\frac{b\mu}{b-\mu}, \left(\frac{rb}{b-\mu}\right) \left(1 - \frac{b\mu}{K(b-\mu)}\right), \frac{b\mu}{b-\mu}, \left(\frac{rb}{b-\mu}\right) \left(1 - \frac{b\mu}{K(b-\mu)}\right)\right) = \begin{bmatrix}
\alpha & \eta & m & 0 \\
\beta & -\delta & \xi & \delta \\
m & 0 & \alpha & \eta \\
\xi & \delta & \beta & -\delta 
\end{bmatrix}$$

Where,

$$\alpha = \frac{-rb^2\mu - rbK\mu + rK\mu^2 + rb\mu^2 + mKb^2 - mKb\mu}{bK(b - \mu)}$$

$$\beta = \frac{(b^2\mu^2 + b^2 + db^2 + 2b^2\mu - 2b\mu^2 - 2b\mu + \mu^2)(Kb - K\mu - b\mu)r}{b(b - \mu + b\mu)^2K}$$

$$\delta = \frac{d(b - \mu)}{b - \mu + b\mu}$$

$$\xi = \frac{(-Kb - K\mu + b\mu)brd}{K(b - \mu + b\mu)^2}$$

$$\eta = -\mu.$$

By using this substitution we are allowed to neatly calculate the determinate when
finding the characteristic polynomial. After we factor the characteristic polynomial we will then replace the substitutions to find the conditions to make \( E_3 \) asymptotically stable. This equilibrium solution then gives the characteristic polynomial,

\[
\lambda^4 + (-2\alpha + 2\delta)\lambda^3 + (-4\alpha\delta - 2\eta\beta - m^2 + \alpha^2)\lambda^2
\]

\[
+(-2\delta\xi\eta + 2\alpha\eta\beta + 2\delta\alpha^2 - 2m\eta\xi - 2m^2\delta - 2\delta\eta\beta)\lambda
\]

\[
+2\alpha\delta\eta\beta - 2m\delta\xi\eta + 2\alpha\delta\xi\eta - \xi^2\eta^2 + \eta^2\beta^2 - 2m\delta\eta\beta = 0.
\]

Using maple to factor this into two quadratics, we obtain

\[
Q_1 = \lambda^2 + (-\alpha - m)\lambda - \xi\eta - \eta\beta
\]

\[
Q_2 = \lambda^2 + (2\delta + m - \alpha)\lambda + \xi\eta - \eta\beta + 2m\delta - 2\alpha\delta.
\]

**Lemma 5.** Let \( \lambda_1 \) and \( \lambda_2 \) be the roots of the quadratic equation

\[
\lambda^2 + A\lambda + B
\]

(i) When \( B < 0 \), \( \lambda_1 \) and \( \lambda_2 \) are both real and have opposite signs.

(ii) When \( B > 0 \) and \( A > 0 \), both \( \lambda_1 \) and \( \lambda_2 \) have negative real parts.

*Proof.* Since \( \lambda^2 + A\lambda + B = (\lambda - \lambda_1)(\lambda - \lambda_2) \) the roots \( \lambda_1 \) and \( \lambda_2 \) satisfy \( \lambda_1 + \lambda_2 = -A \) and \( \lambda_1\lambda_2 = B \). When \( B < 0 \), it is impossible to have a pair of conjugated complex roots. This implies that \( \lambda_1 \) and \( \lambda_2 \) are both real with opposite signs.

Now consider the case where \( A \) and \( B \) are both positive. If \( \lambda_1 \) and \( \lambda_2 \) are real numbers, then they have the same sign (since \( B > 0 \)) and are both negative (since \( A > 0 \)). If \( \lambda_1 \) and \( \lambda_2 \) are a pair of conjugated complex roots, \( \lambda_1 = \alpha + \beta i \) and \( \lambda_2 = \alpha - \beta i \), then \( 2\alpha = -A < 0 \). This means that both roots have negative real
Then from lemma 5 we see that our only concern is that the coefficients on both of the quadratics are all positive. If we have at least one negative coefficient then that leads to instability of the $E_3$ solution. Let,

$$A_1 = (-\alpha - m)$$

$$B_1 = -\xi \eta - \eta \beta$$

$$A_2 = (2\delta + m - \alpha)$$

$$B_2 = \xi \eta - \eta \beta + 2m\delta - 2\alpha\delta.$$  

Looking at $A_1$, one can deduce that for $A_1$ to be greater then zero we need $b^2 - bK + K\mu + \mu b > 0$. This leads to the inequality, $K < \frac{b^2 + \mu b}{b - \mu}$. We can see that the denominator is non-negative since we saw that an existing condition for coexistence equilibrium solution is $b > \mu$. For the $B_1$ coefficient to be positive we need $-\xi \eta - \eta \beta > 0$. After doing some algebra we obtain,

$$\frac{(Kb - K\mu - \mu b)r\mu}{Kb} > 0$$

Thus we can conclude we need the condition, $\frac{bK}{b+K} > \mu$. This is already an existing condition derived from the equilibrium. With the next two coefficients, one would like to break them down in parts involving the restrictions on the other coefficients. Leaving,

$$A_2 = (2\delta + m - \alpha) = 2\delta + A_1 + 2m.$$ 

Here now one would like to break $A_2$ into parts to determine positivity.

1. $A_1$ is clearly positive from earlier.

2. $2m$ is positive, since migration is positive.
3. \(2\delta = \frac{2d(b-\mu)}{b-\mu+\mu b}\) is positive if, \(b > \mu\).

Therefore \(A_2\) is positive since, \(b > \mu > 0\). Lastly one needs to determine the conditions for the \(B_2\) solution. Noting that,

\[
B_2 = \xi \eta - \eta \beta + 2m\delta - 2\alpha \delta = 2\xi \eta + B_1 + 2\delta(A_2 - 2\delta).
\]

Looking at \(B_2\) in parts again we see,

1. \(2\xi \eta = \frac{2(Kb-K\mu-b\mu)brd\mu}{K(b-\mu+b\mu)^2}\)
   - from this, one can deduce we need \(\frac{bK}{b+K} > \mu\)

2. \(B_1\) is positive from earlier reasoning.

3. \(2\delta(A_2 - 2\delta)\) we see that,
   - we need \(A_2 > 2\delta\).
   - this is true since \(A_2 = 2\delta + A_1 + 2m\).

Therefore we can conclude that the stability conditions for \(E_3\) are \(b > \mu\), \(\frac{bK}{b+K} > \mu\), and \(K < \frac{b^2 + \mu b}{b-\mu}\). 

Thus we now have sufficient conditions for stability of the coexistence state of the species. The idea here is that if these conditions are satisfied then the system will be stable and lead to coexistence. On the other hand, if the conditions are not satisfied that does not imply that the system will be unstable. To assure that a system will be unstable, one would the have to draw a different set of conditions leading to instability.

**Theorem 7.** If \(b < \mu\) or \(\frac{bK}{b+K} < \mu\) then the solution

\[
E_3 = \left(\frac{b\mu}{b-\mu}, \left(\frac{rb}{b-\mu}\right) \left(1 - \frac{b\mu}{K(b-\mu)}\right), \frac{b\mu}{b-\mu}, \left(\frac{rb}{b-\mu}\right) \left(1 - \frac{b\mu}{K(b-\mu)}\right)\right)
\]

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is unstable.

Proof. Recall,

\[ Q_1 = \lambda^2 + (-\alpha - m)\lambda - \xi \eta - \eta \beta \]
\[ Q_2 = \lambda^2 + (2\delta + m - \alpha)\lambda + \xi \eta - \eta \beta + 2m\delta - 2\alpha \delta. \]

From Lemma 5 we determined that if the constants for the trinomials were less then zero, that would be a sufficient condition for instability. Let,

\[ B_1 = -\xi \eta - \eta \beta \]
\[ B_2 = \xi \eta - \eta \beta + 2m\delta - 2\alpha \delta. \]

If \( B_1 < 0 \), then \( \lambda_1 \) or \( \lambda_2 \) have opposite signs leading to instability. Thus for \( B_1 = \frac{(Kb-K\mu-\mu b)r\mu}{Kb} < 0 \), one can conclude that a condition that leads to instability of \( E_3 \) is \( \frac{bK}{b+K} < \mu \). If \( B_2 < 0 \), then \( \lambda_3 \) or \( \lambda_4 \) implies the roots have opposite signs. Thus if \( B_2 = \xi \eta - \eta \beta + 2m\delta - 2\alpha \delta = 2\xi \eta + B_1 + 2\delta(A_2 - 2\delta) \). Then need \( 2\xi \eta, B_1, \) and \( 2\delta(A_2 - 2\delta) \) to be less then zero. One then sees a instability condition for \( E_3 \) is \( b < \mu \).

\[ \square \]
The term $\frac{bN_iP_i}{b+N_i}$ in the equations calculates the number of prey who are killed by predators. The numerator calculates a potential number of kills; the denominator adjusts this for the density of prey to reflect the notion that the relationship between the density of prey and the number of prey killed is hyperbolic – that is, it has an upper limit for any number of predators, regardless of how many prey are available.

The numerator calculates the number of pair-wise interactions possible between prey and predators, $NP$. The parameter $b$ reflects the probability that any pair-wise interaction results in a killing (or consumption). The denominator adjusts the resulting quantity for the effects of density [19, 20, 17]. As the population of prey ($N$) increases for any given number of predators ($P$), the number of prey killed per predator will increase at a decreasing rate to some upper limiting value, $bP$. That is, $\lim_{N \to \infty} \frac{bN_iP_i}{b+N_i} = bP$.

Figure 2 shows how the functional response acts in the dynamics of a two species system. We now describe the three types of Holling responses [20, 3].

1. Type I functional response is found in passive predators like spiders. The number of flies caught in the net is proportional to fly density. This is the linear case where the predator increase along with increase population of the prey. To achieve this type of behavior in our model, one would need to change the functional response term from $\frac{bN_iP_i}{b+N_i}$ to $bN_iP_i$. Noticing now that this change makes it a linear function of $N_i$. This means that as the prey population increases, the predator population will increase consumption.
2. Type II functional response is most typical and corresponds to the equation above. Search rate is constant. Plateau represents predator saturation. Prey mortality declines with prey density. Predators of this type cause maximum mortality at low prey density. For example, small mammals destroy most of gypsy moth pupae in sparse populations of gypsy moth. However in high-density defoliating populations, small mammals kill a negligible proportion of pupae. The Holling type II is the functional response applied in this paper.

3. Type III functional response occurs in predators which increase their search activity with increasing prey density. For example, many predators respond to kairomones (chemicals emitted by prey) and increase their activity. Polyphagous vertebrate predators (e.g., birds) can switch to the most abundant prey species by learning to recognize it visually. Mortality first increases with prey increasing density, and then declines. To obtain this type of behavior in our model.
the functional response needs to be changed to \( \frac{bP_i}{1+e^{-bN_i}} \)

Figure 5: Holling Type III

6.1 Modeling functional response

Holling (1959) suggested a model of functional response which remains most popular among ecologists [17]. This model is often called "disc equation" because Holling used paper discs to simulate the area examined by predators. Holling studied predation of small mammals on pine sawflies, and he found that predation rates increased with increasing prey population density [4, 8]. This resulted from 2 effects: (1) each predator increased its consumption rate when exposed to a higher prey density, and (2) predator density increased with increasing prey density. Holling considered these effects as 2 kinds of responses of predator population to prey density: (1) the functional response and (2) the numerical response.
7 NUMERICAL SIMULATIONS

In this section we will graphically display the dynamics and pattern of the populations in the system by utilizing the conditions for stability and instability from section 6. The numerical simulations help us get an understanding of the long term behavior of the system.

7.1 Numerical Simulations for Equilibrium $E_2$.

Figure 6: Stability of $E_2$

Figure 6 shows the stability of equilibrium $E_2$, when $\frac{bK}{b+K} < \mu$. This was derived in Theorem 5. By choosing the parameters to be $r = .65$, $K = 1.9$, $b = .75$, $m = .15$, $d = .15$, and $\mu = .65$ one can then see that the prey populations will go to the equilibrium of $K = 1.9$ and the predator populations will tend to extinction. From Corollary 1 we deduced that $\frac{bK}{b+2K} < \mu$ was a required condition for global exponential stability of the predator. This is directly reflected above in the simulation with the values we have chosen. This means that the death rate chosen for the predator is to large given the carrying capacity of the prey. As well as, the death rate value being dependent on the functional response, meaning we are not allowing enough consumption of prey for the predator. As you will see in Figure 7, we increase the functional response value.
Figure 7 shows the instability of equilibrium $E_2$, when $\frac{bK}{b+K} > \mu$, which was derived in Theorem 5. By choosing the parameters to be $r = .65, K = 1.9, b = 1.9, m = .15, d = .15$, and $\mu = .65$ one can then see that the populations continue to oscillate as $t$ approaches infinity. This means that the populations to not stabilize to approach the equilibrium $E_2$. The long term behavior of this simulation shows the populations stabilizing but not approaching the desired equilibrium. Also one can see that the condition in Corollary 1 for global exponential stability is not satisfied, hence does not lead to extinction of the predator.

7.2 Numerical Simulations for Equilibrium, $E_3$.

We now turn our attention to the last equilibrium solution, $E_3$. This equilibrium solution is going to be the main focus during our numerical simulations. While picking our parameters for the model we need to make sure that we need to pay attention to our conditions. Figure 8 shows the stability of the $E_3$ equilibrium,
by satisfying the conditions for asymptotic stability. Theorem 6 states that the system has a coexistent state when $b > \mu$ and $\frac{bK}{b+K} > \mu$. We also concluded that for stability we needed, $K < \frac{b^2 + \mu b}{b - \mu}$. If we choose the parameters $r = .65, K = 1.9, b = .75, m = .15, d = .15$, and $\mu = .4$ these conditions will be satisfied. One can notice an initial increase of the predator population in patch 2, due to the high number of prey in that patch. After the predators in patch two consume the prey at two high of a rate then both the populations in patch two decline. The consumption and prey cause patch two to decrease population in both species at an alarming rate. Since the populations in patch one are smaller, one can notice a opposite reaction. As they fluctuate over time, they begin to stabilize at equilibrium, $E_3 = \left(\frac{b\mu}{b-\mu}, \frac{rb}{b-\mu} \left(1 - \frac{b\mu}{K(b-\mu)}\right), \frac{b\mu}{b-\mu}, \frac{rb}{b-\mu} \left(1 - \frac{b\mu}{K(b-\mu)}\right)\right) = (.86, .76, .86, .76)$.

Figure 9: Instability of $E_3$

Figure 9 shows the instabilitly of the $E_3$ equilibrium. Theorem 7 states that the system has a coexistent state when $b < \mu$ and $\frac{bK}{b+K} < \mu$. We also concluded that for instability we needed $b < \mu$ or $\frac{bK}{b+K} < \mu$. If we choose the parameters $r = .65, K = 1.9, b = .75, m = .15, d = .15$, and $\mu = .4$ these conditions will be satisfied. The dynamics of this system are erratic and over time the populations do not stabilize. Noticing here that the the migration parameters allow for stability for each species in the patches. Meaning the same number of prey in each patch and same predator in each patch.
In conclusion, we have verified the ultimate bounds for the predator and prey populations. The bounds for the prey population are,

\[ 0 \leq N(t) \leq \left[ \left( \frac{1}{N(0)} - \frac{1}{2K} \right) e^{-rt} + \frac{1}{2K} \right]^{-1} \leq 2K. \]

For the predator populations we found that there was no finite upper bound, represented by the equation,

\[ 0 \leq P(t) \leq (P_1(0) + P_2(0)) e^{(\frac{2bK}{2b+K} - \mu) t}. \]

Since the predators do not have a carrying capacity, there is no restriction on their growth. We were also able to find a lower bound for the predator populations due to easy simplification of the inequalities, obtaining

\[ P(t) \geq (P_1(0) + P_2(0)) e^{-\mu t}. \]

By taking the limit of this result one can see that the lower bound for the predators approaches zero.

Through linearization, by the method of using the Jacobian, we were able to determine some conditions that lead to the equilibrium solutions being stable, asymptotically stable, or unstable. These conditions allowed for a few educated parameters, to plug into the system for numerical simulations. We saw through the plots that the situations provide, closely reflected what the conditions implied.

Further research of this model, one would like to focus in on the functional response. By changing the functional response to have \( \frac{bN_i P_i}{c+Ni} \) we would be adding an extra pa-
rameter \( c \). The denominator adjusts the resulting quantity for the effects of density. As the population of prey increases for any given number of predators, the number of prey killed per predator will increase at a decreasing rate to some upper limiting value. The speed at which this limiting value on per capita consumption is reached (i.e. satiation) is controlled by the parameter \( c \). The greater the value of \( c \), the lower the point of satiation. By adding this parameter, one would also be making the model less symmetric. Less symmetry in the model would lead to more equilibrium solutions, thus allowing more numerical simulation.

Using different Holling type functional responses, is another way to gain further insight on the model. Replacing the Holling type II with type III or type I would allow us to see how different species other than mammals interact. The most complicated version of the model could be seen by using the Holling type III functional response, since this type does not ensure symmetry.
I, Brevin Shae Rock started my college career at Cape Fear Community College. After taking an entry level college course with Jody Hinson, I then found myself wanting to further my education in mathematics. I then transferred to University of North Carolina Wilmington in hopes of majoring in math. One of my earliest most memorable experiences as a mathematics major was with professor Gabriel Lugo. After a poor attempt at my first Calculus II test he told me "you might want to consider changing your major". I was crushed, this comment fueled me to fulfill my ambitions of obtaining my B.S. degree. Aside from this, Dr. Lugo has not only been a great professor, but a positive influence on my physical life. Thank you.

After the incident in my Calculus II class, I went on to succeed in my studies for an undergraduate degree in mathematics. Once I achieved my B.S. in Mathematics at UNCW, I continued my education by doing the masters program at the university as well. While attending the university I held a part-time job as a tutor and an instructor at Cape Fear Community College. Upon completion of the masters program, I obtained a full-time position at the same community college where I started my education. Just goes to show anything is possible with hard work, determination, and a great entourage of family, friends, and professors.
REFERENCES


